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MATHEMATICS COMPETITIONS

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WORLD FEDERATION OF NATIONAL MATHEMATICS COMPETITIONS

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The aims of the Federation are:–

1. ***to promote excellence in, and research associated with, mathematics education through the use of school mathematics competitions;***
2. ***to promote meetings and conferences where persons interested in mathematics contests can exchange and develop ideas for use in their countries;***
3. ***to provide opportunities for the exchanging of information for mathematics education through published material, notably through the Journal of the Federation;***
4. ***to recognize through the WFNMC Awards system persons who have made notable contributions to mathematics education through mathematical challenge around the world;***
5. ***to organize assistance provided by countries with developed systems for competitions for countries attempting to develop competitions;***
6. ***to promote mathematics and to encourage young mathematicians.***

From the Editor

Welcome to *Mathematics Competitions* Vol 16, No 1.

A feature of this issue is the inclusion of some further excellent keynote addresses from the highly successful WFNMC Congress-4 in Melbourne in 2002.

Again, I would like to thank the Australian Mathematics Trust for its continued support, without which the journal could not be published, and in particular Heather Sommariva and Richard Bollard for their assistance in the preparation of the journal.

Submission of articles:

The journal *Mathematics Competitions* is interested in receiving articles dealing with mathematics competitions, not only at national and international level, but also at regional and primary school level. There are many readers in different countries interested in these different levels of competitions.

- The journal traditionally contains many different kinds of articles, including reports, analyses of competition problems and the presentation of interesting mathematics arising from competition problems. Potential authors are encouraged to submit articles of all kinds.

- To maintain and improve the quality of the journal and its usefulness to those involved in mathematics competitions, all articles are subject to review and comment by one or more competent referees. The precise criteria used will depend on the type of article, but can be summarised by saying that an article accepted must be correct and appropriate, the content accurate and interesting, and, where the focus is mathematical, the mathematics fresh and well presented. This editorial and refereeing process is designed to help improve those articles which deserve to be published.

At the outset, the most important thing is that if you have anything to contribute on any aspect of mathematics competitions at any level, local, regional or national, we would welcome your contribution.

Articles should be submitted in English, with a black and white photograph and a short profile of the author. Alternatively, the article can be submitted on an IBM PC compatible disk or a Macintosh disk. The preferred format is LaTeX or TeX, but any text file will be helpful.

Articles, and correspondence, can also be forwarded to the editor by mail to

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or by email to the address <warrena@amt.canberra.edu.au> or by fax to the Australian Mathematics Trust office, + 61 2 6201 5052,
(02 6201 5052 within Australia).

Warren Atkins,
June 2003

* * *

From the President

There are two major items I wish to discuss in this column.

First is an update on ICME-10 in Copenhagen. I did report in the last issue that it appeared there would be some changes in the treatment of Competitions at ICME-10. In previous years, Competitions have had the status of a Topic Group, which allows individual presentations, while WFNMC has been allocated about 2 hours for its own meetings which can be run in its own way. I have now been advised by the organisers of ICME-10 that it is likely that WFNMC and the other Affiliated Study Groups are likely to be allocated three 1-hour slots at which we can have our official meeting and election of a new executive to take us through to 2008, as well as other presentations of our choice. The Topic Group status of Competitions as a topic of general interest has been changed to that of a Discussion Group. In fact this will be DG 16 and Andre Deledicq of Paris and I have been appointed by the International Programme Committee to co-convene this group.

It is true that as a discussion group there will be no presentation of individual papers. However this group will be an important forum for discussion about the issues of Competitions. Andre and I will need to define some form of framework for discussion, but that discussion will be quite free. It will be interactive and people with all points of view will be able to make them during the session.

The IPC is expecting quite strong interest in this group and will provide strong support. The latest information I have is that they are planning on providing the group with a 200 seat auditorium and perhaps some smaller rooms for breakout groups. I strongly urge members to participate enthusiastically in this group. It will be our only normal opportunity to get together until 2006 when we have our own conference in the United Kingdom. With relation to the UK Conference, I am also able to report that our hosts are at present evaluating two venues, but I believe it is best for them to give us their own news updates directly through this journal and I hope that this will commence in the next edition of this Journal.

Before leaving this topic I had speculated that we might stay in

Copenhagen for a day or so before or after ICME-10 to allow more time for us to meet with individual presentations possible. I was planning to visit Copenhagen in June to organise a venue, but for personal reasons I have had to call off this visit at the last moment. This is a pity, but I do believe that we should try to establish this as a tradition at future ICMEs. It is a practice successfully carried out by similar groups such as PME.

The other major item I wish to report on is the ICMI Study 16, for which organisation is now getting under way. Ed Barbeau and I have been appointed Co-Chairs of this Study, entitled 'Challenging Mathematics in and Beyond the Classroom'. It is of course expected that Competitions and related activities, and their role in the education arena, will be main topics, but the study is broader than this and will be looking at methods of challenge beyond the normal interest of WFNMC.

Recently in Toronto, Ed and I met, also with Bernard Hodgson, of Laval University, Quebec City, and Secretary-General of ICMI, to begin planning for this study. ICMI has appointed ten other people to form the International Programme Committee which controls the study. Four of these ten will be familiar to members of WFNMC, namely Patricia Fauring (Argentina), Derek Holton (New Zealand), Ali Rejali (Iran) and Mark Saul (USA). One early feature of the study will be the development of a discussion document. We are hoping that the IPC will meet in Italy in late November to discuss and develop this document.

Eventually there will be a Study Conference, limited to about 80 successful applicants who will demonstrate a high level of contribution and activity. This conference may be held in Brisbane, Australia, in 2006, and from it a book will be published, possibly by Kluwer, which will be the official findings of the study.

I will ensure that the WFNMC website

www.amt.canberra.edu.au/wfnmc.html

will establish a section on the study and provide up to date information on its progress. In the meantime, I urge members of WFNMC to take a vital interest in it, and, if possible, plan to be one of those participants at the study conference. The Study will have great significance for us and articulate the role of competitions and other challenges in the world of mathematical education in a way we have been hoping for some considerable time.

Cheers

Peter Taylor
Canberra
June 2003

* * *

The 50th Anniversary of One Problem: The Chromatic Number of the Plane & Its Relatives Part 1[1]

Alexander Soifer

Alexander Soifer is Professor at the University of Colorado. He is chair and founder of the Colorado Mathematical Olympiad, now in its nineteenth year, a member of USA Mathematics Olympiad Subcommittee, Secretary of the World Federation of National Mathematics Competitions and Editor of the Quarterly, Geombinatorics. Soifer's publications include four books, with a fifth currently in preparation.

1. The Problem

Perhaps, mathematics is at her best, when anyone can understand a problem, yet nobody could conquer it. Today we will discuss precisely such a problem. It has withstood all attempts for over 50 years. Here it is:

> *What is the smallest number of colors with which we can color the plane in such a way that no color contains a monochromatic segment of length 1?*

This number is called the chromatic number of the plane and is often denoted by χ. (The term monochromatic segment simply stands for a pair of points of the same color.)

[1]This paper was presented as a keynote address at WFNMC Congress-4, Melbourne August 2002

Please note that here we do not necessarily have 'nice' maps with closed regions. We assign one color to every point of the plane without any restrictions. I do not know who first noticed the following result. Perhaps, Adam or Eve? To be a bit more serious, I do not think that ancient Greek geometers, for example, knew this nice fact. They just did not ask such questions!

Problem 1. (Adam & Eve?). Any 2-colored plane contains a monochromatic segment of length 1, i.e.,

$$\chi \geq 3$$

Solution. Toss on the given 2-colored plane an equilateral triangle T of side 1 (Figure 1). We have only 2 colors while T has 3 vertices (I trust you have not forgotten the Pigeonhole Principle). Two of the vertices must lie on the same color. They *are* distance 1 apart.

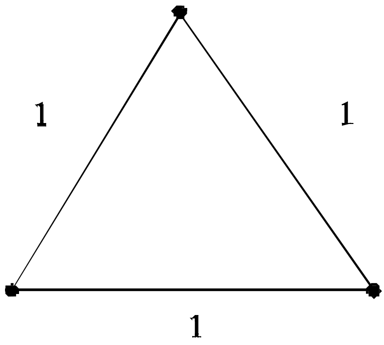

Figure 1

Problem 2. Any 3-colored plane contains a monochromatic segment of length 1, i.e.,

$$\chi \geq 4$$

Solution By the Canadian geometers, brothers Leo and William Moser, (1961, [30]). Toss on the given 3-colored plane what we now call *The Moser Spindle* (Figure 2). Every edge in the spindle has the length 1.

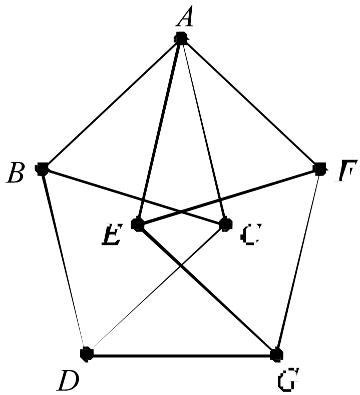

Figure 2. The Moser Spindle

Assume that the seven vertices of the spindle do not contain a monochromatic segment of length 1. Call the colors used to color the plane red, white, and blue. The solution now will faithfully follow the childrens' song: 'A B C D E F G...'.

Let the point A be red, then B and C must be one white and one blue, therefore D is red. Similarly E and F must be one white and one blue, therefore G is red. We got a monochromatic segment DG of length 1 in contradiction to our assumption.

Does an upper bound for χ exist? It is not immediately obvious. Can you find one? It is natural to try a regular tiling — and indeed, one works!

Problem 3. There is a 7-coloring of the plane that contains no monochromatic segments of length 1.

$$\chi \leq 7$$

Solution ([19]). We can tile the plane by regular hexagons of side 1. Now we color one hexagon in color 1, and its six neighbors in colors 2, 3,..., 7 (Figure 3). The union of these seven hexagons forms a flower, a symmetric polygon P of 18 sides. Translates of P (i.e., images of P

under translations) tile the plane and determine how we color the plane in 7 colors.

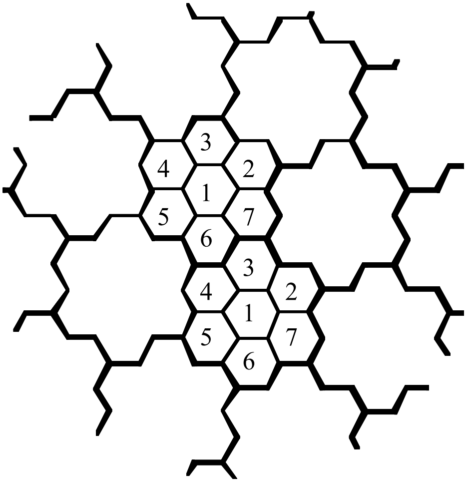

Figure 3

It is easy to compute (please, do) that each color does not have monochromatic segments of any length d, where $2 < d < \sqrt{7}$.

Thus, if we shrink all linear sizes by a factor of, say, 2.1, we will get a 7-coloring of the plane that has no monochromatic segments of length 1. (Observe: due to the above inequality, we have enough cushion, so that it does not matter in which of the two adjacent colors we color boundaries).

In 1982 the Hungarian mathematician Laszlo Szekely found a way to prove the upper bound without using hexagonal tiling. He used tiling by squares. Can you think of Laszlo's proof? You can find it, as well as an alternative proof of the lower bound, in [39].

It is amazing that pretty easy problems 2 and 3 give us the best known to mathematics bounds for χ. They were published over 40 years ago (in fact, they are older than that: see the next section for an historical account). Still, all we know is that

$$\chi = 4, \text{ or } 5, \text{ or } 6, \text{ or } 7$$

A very broad spread! Which do you think it is? Paul Erdós told me that he was sure $\chi = 5$.

Recently the renowned American geometer Victor Klee shared with me a very interesting story. In 1980-1981 he lectured in Switzerland. The celebrated Swiss mathematician Van der Waerden was in attendance. When Professor Klee presented the state of this problem, Van der Waerden became very interested. Right there, during the lecture he started working on the problem. He tried to prove that $\chi = 7$!

What little time I spent so far working on this problem makes me feel that $\chi \geq 6$. Paul Erdös believed that 'God has a transfinite Book, which contains all theorems and their best proofs, and if He is well intentioned toward those, He shows them the Book for a moment' [10]. If I ever deserved the honor and had a choice, I would ask to peek at the page with the chromatic number of the plane. Wouldn't you!

2. The History

It is natural for one to inquire into the authorship of one's favorite problem. And so I turned to tons of articles and books. Some of the information I found appears here in table 4. Are you confused? I was too!

As you can see in the table, Douglas Woodall credits Martin Gardner, who in turn refers to Leo Moser. Hallard Croft calls it 'a long standing open problem of Erdös', while Paul Erdös cannot trace the origin of this problem'. Later Erdös credits 'Hadwiger and Nelson', while Vic Klee and Stan Wagon write that the problem was 'posed in 1960-61 by M. Gardner and Hadwiger'. Croft comes again, this time with Kenneth Falconer and Richard Guy, to cautiously suggest that the problem is 'apparently due to E. Nelson [CFG]. Yet, Richard Guy could not tell me who 'E. Nelson' was and why Guy & Co. 'apparently' attributed the problem to him (conversation in a car in Keszthely, Hungary, 1993).

Thus, at least five mathematicians were credited with the problem: Paul Erdös, Martin Gardner, Hugo Hadwiger, Leo Moser, and Edward Nelson.

What a great group of mathematicians! But it was hard for me to believe that they all created the problem, even independently. I felt like a private investigator untangling a web of conflicting accounts. Six months later I solved the puzzle! I would like to thank a great number of mathematicians for contributing their part of the puzzle. I am especially grateful to Paul Erdös, Victor Klee, Martin Gardner, Edward Nelson and John Isbell. Only their accounts, recollections and congeniality made these findings possible.

Publication	Year	Author(s)	Problem Creator(s) or Source Named
[15]	1960	Gardner	'Leo Moser ...writes...'
[19]	1961	Hadwiger (after Klee)	Nelson
[5]	1961	Erdös	'I cannot trace the origin of this problem'
[11]	1967	Croft	'A long standing open problem of Erdös'
[40]	1973	Woodall	Gardner
[7], [8], [9]	1980-1981	Erdös	Hadwiger and Nelson
[2]	1991	Croft, Falconer and Guy	'Apparently due to E. Nelson'
[29]	1991	Klee and Wagon	'Posed in 1960-61 by M. Gardner and Hadwiger'

Table 4. Who created Chromatic Number of the Plane Problem?

The problem creator was born on May 4, 1932 (a good number: 5/4/32), in Decatur, Georgia. The son of the secretary of the Italian YMCA, Joseph Edward Nelson had studied at a liceo (Italian prep school) in Rome. In 1949 Ed had returned to the U.S. and entered the University of Chicago. The visionary President of the University, Robert Hutchins, allowed students to avoid 'doing time' at the University by passing lengthy placement exams. Ed Nelson had done so well that he was allowed to go right on to a Master's program without working on his Bachelor's.

Time magazine reported young Nelson's fine achievements in 14 (!) exams on December 26, 1949, next to the report on completion of the last war-crimes trials of World War II (Field Marshal Fritz Erich von

Manstein received 18 years in prison), assurances by General Dwight D. Eisenhower that he would not be a candidate in the 1952 Presidential election (he certainly was), and a promise to announce *Time's* 'A Man of the Half-Century' in the next issue. (Consequently *Time's* choice was Winston Churchill.)

The following fall the 18-year-old Edward Nelson created what he called 'a second four-color problem' (from his October 5, 1991, letter [32] to me):

> 'Dear Professor Soifer:
>
> In the autumn of 1950, I was a student at the University of Chicago and among other things was interested in the four-color problem, the problem of coloring graphs topologically embedded in the plane. These graphs are visualizable as nodes connected by wires. I asked myself whether a sufficiently rich class of such graphs might possibly be subgraphs of one big graph whose coloring could be established once and for all, for example, the graph of all points in the plane with the relation of being unit distance apart (so that the wires become rigid, straight, of the same length, but may cross). The idea did not hold up, but the other problem was interesting in its own right and I mentioned it to several people.'

One of these people was John Isbell. He still remembers it very vividly (from his August 26, 1991, letter [26] to me):

> '...Ed Nelson told me the problem and $\chi \geq 4$ in November 1950, unless it was October—we met in October. I said what upper bound have you, he said none, and I worked out 7. I was a senior at the time (B.S., 1951). I think Ed had just entered U. Chicago as a nominal sophomore and taken placement exams which placed him a bit ahead of me, say a beginning graduate student with a gap or two in his background. I certainly mentioned the problem to other people between 1950 and 1957; Hugh Spencer Everett III, the author of the many-worlds interpretation of quantum mechanics, would certainly be one, and Elmer Julian Brody who did a doctorate under Fox and has long been at the Chinese University of Hong Kong and is

> said to be into classical Chinese literature would be another. I mentioned it to Vic Klee in 1958 ±1..."

Victor Klee too remembers (our phone conversation, 1991) hearing the problem from John Isbell in 1957-1958. In fact, it took place before September 1958 when Professor Klee left for Europe. There he passed it to Hugo Hadwiger who was collecting problems for the book *Open Problems in Intuitive Geometry* to be written jointly by Erdös, Fejes-Toth, Hadwiger, and Klee (this great book-to-be has never materialized).

What are the roles of Paul Erdös, Martin Gardner, and Leo Moser in our story of the problem? The only question I am leaving for others to answer is how and when Leo Moser came by the problem. Apparently he did not create it independently from Edward Nelson (Paul Erdös' July 16, 1991, letter [12] to me):

> 'I do not remember whether Moser in 1958 told me how he heard the problem on the chromatic number of the plane, I only remember that it was not his problem.'

Yet, Leo Moser made a valuable contribution to the survival of this problem. He gave it to both Paul Erdös and Martin Gardner. Martin Gardner, due to his impeccable taste, recognized the value of this problem and included it in his 'Mathematical Games' column in *Scientific American* ([16]), with the acknowledgement that he received it from Leo Moser of the University of Alberta. Thus, the credit for the first publication of the problem goes to Martin Gardner. It is beyond me why so many authors of articles and books, as old as 1973 and as recent as 1991, give the credit for the creation of the problem to Martin Gardner, something he himself has never claimed. (In our 1991 phone conversation Martin told me for a fact that the problem was not his, and he promptly listed Leo Moser as his source.)

Moreover, some authors (Victor Klee and Stanley Wagon, for example) who knew of Nelson, still credited Gardner and Hadwiger because they would accept only written, preferably published word. Following this logic, the creation of the celebrated Four-Color Map Coloring Problem must be attributed to Augustus De Morgan (who first wrote about it in his 1852 letter), or better yet to Arthur Cayley (whose abstract was first publication on the problem). Yet we all seem to agree that the twenty-year-old Francis Guthrie created this problem, even though he did not publish a word about it!

Of course, a lone self-serving statement would be too weak a foundation for a historical claim. On the other hand, independent selfless testimonies supporting each other comprise as solid a foundation for a historical claim as any publication. And this is precisely what my inquiry has produced. Here is just one example of Nelson and Isbell's selflessness. Edward Nelson wrote to me on August 23, 1991 [31]:

> 'I proved nothing at all about the problem ...'

John Isbell corrected Nelson in his September 3, 1991, letter [Isb2] to me:

> 'Ed Nelson's statement which you quote, 'I proved nothing at all about the problem,' can come only from a failure of memory. He proved to me that the number we are talking about is * 4, by precisely the argument in Hadwiger 1961. Hadwiger's attribution (on Klee's authority) of that inequality to me can only be Hadwiger's or Klee's mistake."

This brings us to the issue of authorship of two inequalities

$$4 \leq \chi \leq 7$$

Once again the entire literature is off the mark by giving credit for first proofs to Hadwiger and the Mosers. Yes, in 1961 the famous Swiss geometer Hugo Hadwiger published ([19]) the chromatic number of the plane problem together with proofs of both inequalities. But he writes (and nobody reads!):

> 'We thank Mr. V. L. Klee (Seattle, USA) for the following information. The problem is due to E. Nelson; the inequalities are due to J. Isbell.'

Hadwiger does go on to say: 'some years ago the author (i.e., Hadwiger) discussed with P. Erdös questions of this kind.' Does he mean that he thought of the problem independently from Nelson? We will never find out for sure, but I have my doubts. Hadwiger jointly with H. Debrunner published an excellent, long problem paper in 1955 [18] that was extended to their wonderful book in 1959 [19]; see also the 1964 English translation [23] with Victor Klee, and the 1965 Russian translation [22] edited by Isaac Yaglom. All these books (and Hadwiger's other papers) include a number of 'questions of this kind', but do not include the chromatic number of the plane problem. Also, it seems to me that the problem in question is somewhat out of Hadwiger's 'character': in all similar problems he prefers to consider closed rather than arbitrary sets.

I shared my doubts about Hadwiger independently creating the problem with Paul Erdös. It was especially important because Hadwiger mentioned Erdös as his witness of sorts (please, see the previous paragraph). Paul replied (Erdös July 16, 1991, letter [12] to me) as follows:

> 'I met Hadwiger only after 1950, thus I think Nelson has priority (Hadwiger died a few years ago, thus I cannot ask him, but I think the evidence is convincing).'

During his talk at 25th South Eastern International Conference On Combinatorics, Computing and Graph Theory in Boca Raton, Florida at 9:30-10:30 AM on March 10, 1994, Paul summarized the results of my research in uniquely Erdösian style:

> 'There is a mathematician called Nelson who in 1950 when he was an epsilon, that is he was 18, discovered the following question. Suppose you join two points in the plane whose distance is 1. It is an infinite graph. What is chromatic number of this graph? Now, de Bruijn and I showed that if an infinite graph which is chromatic number k, it always has a finite subgraph, which is chromatic number k. So this problem is really [a] finite problem, not an infinite problem. And it was not difficult to prove that the chromatic number of the plane is between 4 and 7. I would bet it is bigger than 4, but I am not sure[2]. And the problem is still open. If it would be my

[2] In 1980 Paul was more positive [7]: 'I am sure that $a_2 > 4$, but cannot prove it.' (a_2 here stands for the chromatic number of the plane)

> problem, I would certainly offer money for it. You know, I can't offer money for every nice problem because I would go broke, immediately. I was asked once what would happen if all your problems would be solved, could you pay? Perhaps, not, but it doesn't matter. What would happen to the strongest bank if all the people who have money there would ask for money back? Or what would happen to the strongest country if they suddenly ask for money? Even Japan or Switzerland would go broke. You see, Hungary would collapse instantly. Even the United States would go broke immediately. [snip] Actually it was often attributed to me, this problem. It is certain that I had nothing to do with the problem. I first learned the problem, the chromatic number of the plane, in 1958, in the winter, when I was visiting [Leo] Moser. He did not tell me from where this nor the other problems came. It was also attributed to Hadwiger but Soifer's careful research showed that the problem is really due to Nelson.'

The two famous Canadian problem people, the brothers Leo and William Moser, also published in 1961 [30] the proof of the lower bound $4 \leq \chi$ while solving a different problem. Although in my opinion, these two proofs are not distinct, the Mosers' emphasis on a finite set, now called the Moser Spindle, proved to be very productive.

Now we can finally give a due credit to Edward Nelson for being first in 1950 to prove the lower bound $4 \leq \chi$. Because of this bound, John Isbell recalls in his letter [26] to me, Nelson 'liked calling it a second Four-Color Problem!'

Professor Edward Nelson is now on the faculty at Princeton University; his main area of interest is analysis. A few years ago he was elected into the National Academy of Sciences.

John Isbell was first in 1950 to prove the upper bound $\chi \leq 7$. He used the same hexagonal 7-coloring of the plane that Hadwiger published in 1961 [19]. Please note that Hadwiger first used the same coloring in 1945 [18], but for a different problem: his goal was to show that there are seven congruent closed sets that cover the plane (he also proved there that no five congruent closed sets cover the plane). Professor John Isbell is on the faculty at the State University of New York at Buffalo.

Paul Erdös' contribution to the history of this problem is two-fold. First of all, as Augustus De Morgan did for the Four-Color Problem, Erdös kept the flame of the problem lit. He made the chromatic number of the plane problem well known by posing it in his countless problem talks and many publications, for example [5], [6], [7], [9], [8] and [14].

Secondly, Paul Erdös created a good number of fabulous related problems. We will discuss one of them in the following section.

2. Polychromatic Number of the Plane

When a great problem withstands all assaults, mathematicians create many related problems. It gives them something to solve. Sometimes there is a real gain in this process, when an insight into a related problem brings new ways to conquer the original one. Numerous problems were posed around the chromatic number of the plane. I would like to share with you my favorite among them.

It is convenient to say that a colored set S *realizes distance* d if S contains a monochromatic segment of length d.

Our knowledge about this problem starts with the celebrated 1959 book by Hugo Hadwiger ([21], and consequently its translations [22] and [23]). Hadwiger reported in the book that he had received a 9/9/1958 letter from the Hungarian mathematician A. Heppes:

> 'Following an initiative by P. Erdös he [i.e. Heppes] considers decompositions of the space into disjoint sets rather than closed sets. For example, we can ask whether proposition 59 remains true in the case where the plane is decomposed into three disjoint subsets. As we know, this is still unresolved.'

In other words, Paul Erdös asked whether it was true that if the plane is partitioned into three disjoint subsets, one of the subsets must realize all distances. Soon the problem took on its current 'appearance.' Here it is:

What is the smallest number of colors for coloring the plane in such a way that no color realizes all distances?

This number had to have a name, and so in 1992 [35] I named it the polychromatic number of the plane and denoted it by χ_p. The name and the notation seemed so natural, that by now it has become a standard, and has appeared in such important books as [28] and [17].

Since I viewed this to be a very important open problem, I asked Paul Erdös to verify his authorship, suggested in passing by Hadwiger. As always, Paul was honest and unassuming in his July 16, 1991 letter to me [12]:

> 'I am not even quite sure that I created the problem: Find the smallest number of colors for the plane, so that no color realizes all distances, but if there is no evidence contradicting it we can assume it for the moment.'

In the chromatic number problem we were looking for colorings of the plane such that each color does not realize the distance 1. In the polychromatic number problem we are coloring the plane in such a way that each color i does not realize a distance d_i. For distinct colors i and j, the corresponding non-realizable distances d_i and d_j may (but do not have to) be distinct. Of course,

$$\chi_p \leq \chi.$$

Due to problem 3,

$$\chi_p \leq 7.$$

Nothing else was discovered during the first 12 years of this problem. Then in 1970, Dmitry E. Raiskii from the Moscow High School (!) for Working Youth #105 published ([33]) the lower and upper bounds for χ_m:

Problem 4. ([33]).

$$4 \leq \chi_p \leq 6.$$

The example proving the upper bound was found by S. B. Stechkin and published with his permission by D. E. Raiskii in [33].

Incidentally, Stechkin has never gotten a credit in the West for his example. Numerous articles and books I have read give the credit to Raiskii (except for Raiskii himself!). Why did it happen? As everyone else, I read the English translation of Raiskii's paper [33]. It said:

> '*S. B. Stechkin noted* that the plane can be decomposed into six sets such that all distances are not realized in any one of them. A corresponding example is presented here with the *author's solution*.'

Author of what? – I was wondering. Author of the paper (as everyone decided)? But there is very little need for a 'solution' once the example is found. A criminalistic instinct overcame me. I ordered a copy of the original Russian text. I read it in disbelief:

> 'A corresponding example is presented here with the author's permission.'

Stechkin permitted Raiskii to publish Stechkin's example! The translator mixed up somewhat similarly looking Russian words and 'innocently' created a myth:

Russian word	English translaion
RE XENIE	solution
RAZRE XENIE	permission

Table 5. Translator's Folly.

Three years later, in 1973 the British mathematician D. R. Woodall published one of the best to date papers [40] on problems related to the chromatic number of the plane. Among other things, he gave his own proofs of the inequalities of problem 4. I prefer the lower bound proof by Woodall and the upper bound example by S. B. Stechkin. Let us look at them.

The Woodall proof is based on a triple use of two ideas of Hugo Hadwiger ([23], Problem 54 and 59). I do not like to use the word 'lemma', it is Greek to me. Especially since there is an appropriate English word 'tool'.

Tool 4. Let a circle C of diameter d be colored white and blue. If the white color does not realize the distance dw, $(dw \leq d)$, and the blue color does not realize d_b $(d_b \leq d)$, then C has no monochromatic chord of length d_w.

Solution. The circle C certainly does not have a white monochromatic chord of length d_w, (d_w is not realized by the white color). Assume that C contains a blue monochromatic chord XY of length d_w. Let us rotate XY about the center of C to its new position $X'Y'$, such that $|XX'| = d_b$ (Figure 6).

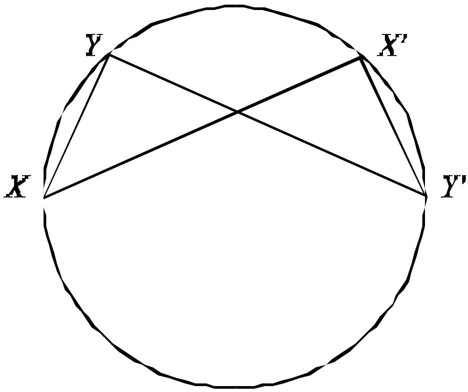

Figure 6

Since X' and Y' may not be both white (or XY' would be a white monochromatic chord of length d_w), at least one of the chords XX', YY' of length d_b is a blue monochromatic chord. A contradiction to the blue color not realizing the distance d_b.

Tool 5. Let the plane be colored red, white, and blue, and each color does not realize a distance, respectively d_r, d_w, d_b; $dr \leq dw \leq db$. Then the plane has no segment of length d_{rb} with one red and one blue endpoints, where

$$d_{rb} = \sqrt{d_b^2 - \left(\frac{1}{2}d_r\right)^2} + \frac{\sqrt{3}}{2}d_r.$$

Solution. Let RB be a segment of length d_{rb} with one endpoint R red, and the other endpoint B blue. We draw a circle C of radius d_b with center in B, and an equilateral triangle T of side d_r with one vertex in R and an altitude on the segment RB, (Figure 7). It is easy to verify (do) that the other two vertices R_1 and R_2 of T lie on the circle C (in fact, the distance d_{rb} was chosen for that purpose).

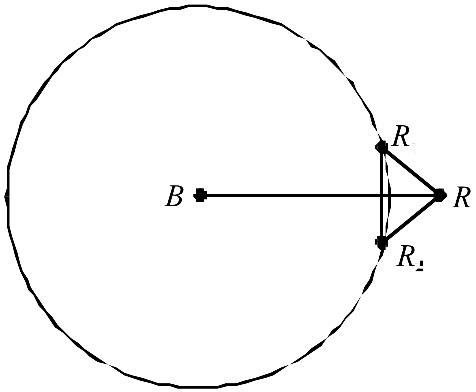

Figure 7

Since the plane has no blue monochromatic segment of length db and the center B of the circle C is blue, the entire circle C is colored red and white. Due to tool 4, C has no monochromatic chord of length dr. Therefore, one of the end-points of the chord R_1R_2 is red. Thus, two of the vertices of the equilateral triangle RR_1R_2 of side d_r are red in contradiction to the red color not realizing the distance d_r.

Now that you have proved tools 4 and 5, we are ready to prove the lower bound inequality.

Problem 6. (D.E. Raiskii [33], D.R. Woodall [40]).

$$\chi_m \geq 4$$

i.e., in any 3-colored plane at least one of the colors realizes all distances.

Solution by D. R. Woodall. Assume the plane is colored red, white, and blue, and in contradiction to the problem statement each color does not realize a distance, respectively d_r, d_w, d_b; $d_r \le d_w \le d_b$. We can assume without loss of generality that each color is present (if a color is not present, just pick one point and repaint it into this color). We define d as follows (d_{rb} is defined in tool 5):

$$d = \sqrt{d_{rb}^2 - \left(\frac{1}{2}d_w\right)^2} \; + \; \frac{\sqrt{3}}{2}d_w.$$

Let us prove that if one endpoint of a segment of length d is white, then the other endpoint is white as well.

Indeed, let BW be a segment of length d with one endpoint B blue and the other W white. We draw a circle C of radius d_{rb} with the center in B and an equilateral triangle T of side d_w with one vertex in W and an altitude on the segment BW (Figure 8). It is easy to check (do!) that the other two vertices W_1 and W_2 of the triangle T lie on the circle C (in fact, the distance d was chosen to guarantee exactly that!).

Due to tool 5, the plane has no segment of length d_{rb} with one red and one blue endpoint. Therefore, the circle C has no red points (its center B is blue). Thus, C is colored white and blue. By tool 4, any chord of C of length d_w has one white endpoint. It means that W_1 or W_2 is white, i.e., two vertices of the equilateral triangle WW_1W_2 of side d_w are white in contradiction to the white color not realizing the distance d_w.

Similarly, we can show that a segment RW of length d with one endpoint R red and the other W white may not exist in the plane. (Just construct a picture like Figure 8, except the center of the circle will be in red R.)

Thus, we proved that a circle of radius d with center in a white point must be completely white. But this means that the entire plane is white. This contradiction completes the proof.

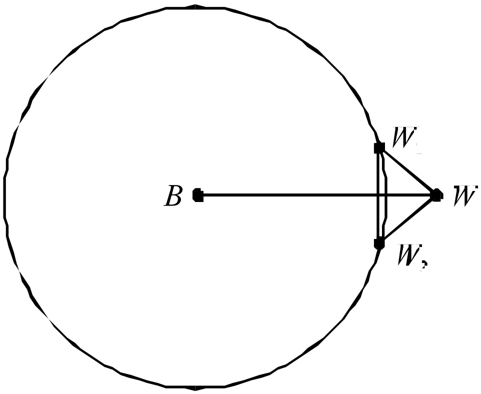

Figure 8

Problem 7. (S. B. Stechkin, [33]).

$$\chi \leq 6$$

i.e., there is a 6-coloring of the plane such that no color realizes all distances.

Solution by S. B. Stechkin, presented by D. E. Raiskii in [33]. The unit of construction is a parallelogram that consists of four regular hexagons and eight equilateral triangles, all of side length 1 (Figure 9). We color the hexagons in colors 1, 2, 3, and 4. There are two types of triangles: to a triangle with a vertex below its horizontal base we assign the color 5, and to a triangle with a vertex above the base we assign the color 6. While coloring, we consider every hexagon to include its entire boundary except its one rightmost and two lowest vertices, and every triangle does not include any of its boundary points.

Now we can tile the entire plane with translations of the unit parallelogram. We are done.

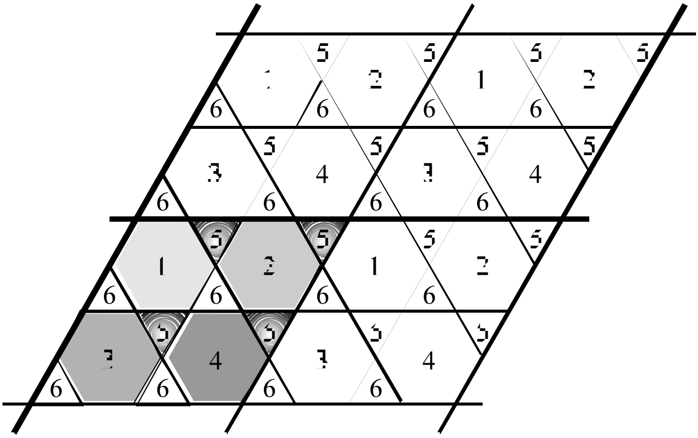

Figure 9

A simple construction solved problem 7. Simple, after it is found. The trick was to find it, and S. B. Stechkin found it first. Christopher Columbus too 'just ran into' America! I got hooked. I found myself thinking about six-coloring of the plane. I felt that if our ultimate goal was to find the chromatic number of the plane χ or to at least improve the known bounds ($4 \le \chi \le 7$), it may be worthwhile to somehow evaluate how close a coloring of the plane is to achieving that. In 1992 I introduced such a measurement.

Definition 8. (A. Soifer [36]). Given an $n-$coloring of the plane such that the color i does not realize the distance d_i ($1 \le i \le n$). Then we would say that this coloring has *type* $(d_1, d_2, \ldots d_n)$.

It would be a great improvement in the chromatic number problem to find a 6-coloring of type (1,1,1,1,1,1), or to show that one does not exist. With the appropriate choice of a unit, we can make the 1970 Stechkin coloring to have type $(1, 1, 1, 1, \frac{1}{2}, \frac{1}{2})$. D. R. Woodall [40] reached his goal of finding a closed 6-coloring of the plane with all distances not realized by any color. His example, however, had three ‘missing distances.’ It had type $(1,1,1,\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{2\sqrt{3}})$.

In a search for a ‘good’ coloring, I looked at a tiling with regular octagons and squares that I saw in many Russian public toilets (Figure 10).

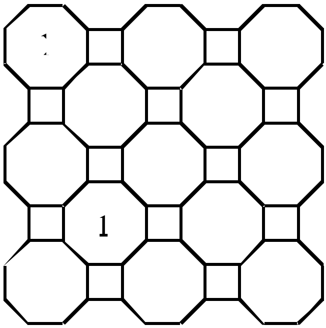

Figure 10

But it didn’t work! See it for yourself:

Problem 9. Prove that the set of all squares in the tiling of figure 10 (even without their boundaries) realizes all distances.

I decided to shrink the squares until their diagonal became equal to the distance between two closest squares. Simultaneously (!) the diagonal of the now non-regular octagon became equal to the distance between the two octagons marked with 1 in Figure 10. I was in business!

Problem 10. (A. Soifer [36]). There is a 6-coloring of the plane of type $(1,1,1,1,1,\frac{1}{\sqrt{5}})$.

Solution. We start with two squares, one of side 2 and the other of diagonal 1 (Figure 11). We can use them to tile the plane with squares and (non-regular) octagons (Figure 13). Colors 1, ..., 5 will consist of octagons; we will color all squares in color 6. With each octagon and each square we include half of its boundary (bold lines in Figure 12) without the endpoints of that half. It is easy to verify (please do!) that $\sqrt{5}$ is not realized by any of the colors 1, ..., 5; and 1 is not realized by the color 6. By shrinking all linear sizes by a factor of $\sqrt{5}$, we get a 6-coloring of type (1, 1, 1, 1, 1, $\frac{1}{\sqrt{5}}$).

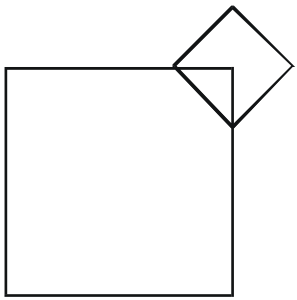

Figure 11

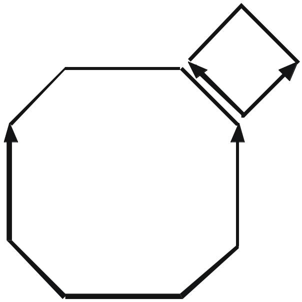

Figure 12

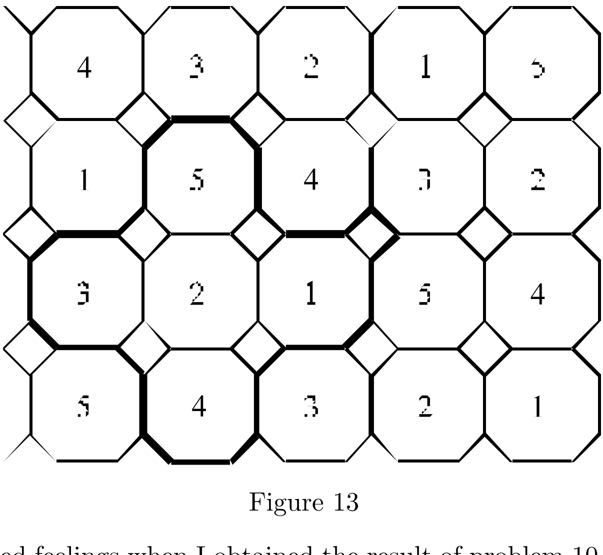

Figure 13

I had mixed feelings when I obtained the result of problem 10 in August 1992. On the one hand, I knew the result was 'close but no cigar': after all, a 6-coloring of type (1,1,1,1,1,1) has not been found. On the other hand, I began to feel that the latter 6-coloring probably did not exist, and if so, my 6-coloring was the best possible. There was another consideration as well. While in a Ph.D. program in Moscow I hoped to produce the longest paper that would still be refereed in by a major journal (and I got one published in 1973 that, in manuscript, was 56 pages long). This time I was concerned with a 'dual record': how short can a paper be and still contain enough to be refereed in and published? The paper solving problem 10 was 1.5 pages long, plus a page of pictures. It was refereed within one day [36]. It also gave birth to a new open problem.

Definition 11. (A. Soifer [24]). *Almost chromatic number* χ_a of the plane is the minimal number of colors that are required for coloring the plane so that almost all (i.e., all but one) colors forbid unit distance, and the remaining color forbids a distance.

We have the following inequalities for χ_a:

$$4 \leq \chi_a \leq 6$$

The lower bound follows from Dmitry Raiskii's [33]. I proved the upper bound in problem 10 above [36]. And the problem is:

Open Problem 12. (A. Soifer [24]). Find χ_a.

4. Continuum of 6–colorings

In 1993 another 6-coloring was found by Ilya Hoffman and the author ([24], [25]). It's type was (1,1,1,1,1,1, $\sqrt{2}-1$). The story of this discovery is noteworthy. In the summer 1993 I was visiting my Moscow cousin, a well-known New Vienna School composer Leonid Hoffman. His 15-year old son Ilya studied violin at the Gnesin's Music High School. Ilya set out to find what I was doing in math, and did not accept any general answers. He wanted particulars. I showed him my 6-coloring (problem 10). Ilya got busy. The next day he showed me . . . the Stechkin coloring that he discovered on his own! Shortly he came up with a idea of using a 2-square tiling. Ilya had an intuition of a virtuoso fiddler and no mathematical culture—I figured out what sizes squares must have for the 6-coloring to do the job we needed. And the joint work of an unusual mathematician-musician pair was born.

Problem 13. (I. Hoffman and A. Soifer [24], [25]). There is a 6-coloring of the plane of type (1,1,1,1,1, $\sqrt{2}-1$)

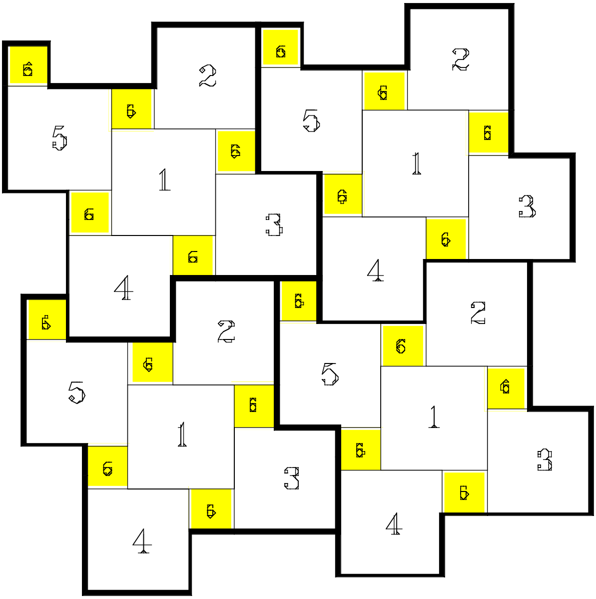

Figure 14

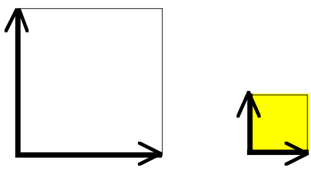

Figure 15

Solution. We tile the plane with squares of diagonals 1 and $\sqrt{2}-1$ (Figure 14). We use colors 1,...,5 for larger squares, and color 6 for all smaller squares. With each square we include half of its boundary, the left and lower sides, without the endpoints of this half (Figure 15).

The two examples of problems 10 and 13 suggest the following open problem.

Open Problem 14. (A. Soifer [37], [38]). Find the *6-realizable set* X_6 of all positive numbers α such that there exists a 6-coloring of the plane of type (1,1,1,1,1, α).

In this new language, problems 10 and 13 can be written as follows:

$$\frac{1}{\sqrt{5}} . \sqrt{2} - 1 \in X_6$$

What do 6-colorings of problems 10 and 13 have in common? It is not obvious, is it? After a while I realized that they are two extreme examples of a general case, and a much better result, in fact, was possible.

Theorem 15. (A. Soifer [38]). $\left[\sqrt{2} - 1, \frac{1}{\sqrt{5}}\right] \subseteq X_6$

i.e. for every $\alpha \in \left[\sqrt{2} - 1, \frac{1}{\sqrt{5}}\right]$ there is a 6-coloring of type (1,1,1,1,1,1, α).

Proof. Let us look at the following tiling of the plane (Figure 16). It is generated by a large square and a small square with the angle C between them (Figure 18). We are ready to color the tiling in Figure 1 in 6 colors. Denote by â the figure bounded by a bold line. Use colors 1 through 5 for octagons inside â and color 6 for all small squares. Include in the colors of octagons and squares the part of their boundaries that is indicated in bold in Figure 17. We wish to guarantee that each color forbids a distance.

Let the side length of the large square be 1, and the segments of adjacent sides of large square inside the small square be denoted by x and y, $x \leq y$ (Figure 18). It is easy to see (Figure 19), that the extensions of sides of the four large squares around one small square cut the latter into four congruent right triangles with sides x and y and a square of side $y - x$.

The conditions for each color to forbid a distance produce the following system of two inequalities (Figure 20):

$$\begin{cases} d_1 \geq d_2 \\ d_3 \geq d_4 \end{cases}$$

Figures 19 and 20 allow an easy representation of all d_i, $(i = 1, 2, 3, 4)$ in terms of x and y. As a result, we get the following system of inequalities:

$$\left.\begin{array}{c} \sqrt{(1+y-x)^2 + (2x)^2} \geq \sqrt{1 + (1-2x)^2} \\ 1 - x - y \geq \sqrt{2(x^2+y^2)} \end{array}\right\}$$

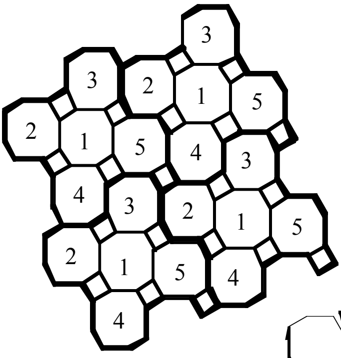

Figure 16

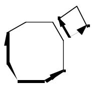

Figure 17

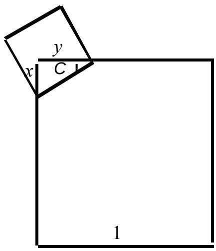

Figure 18

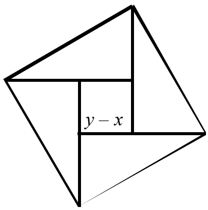

Figure 19

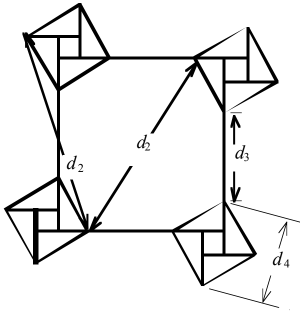

Figure 20

Solving for x each of these two inequalities separately, we get a *unique* solution for x as a function of y that satisfies both of the above inequalities:

$$x = \sqrt{2-4y} + y - 1 \text{ where } 0 \le y \le 0.5 \qquad (1)$$

Since $0 \le x \le y$, we get narrower bounds for $y:\ 0.25 \le y \le \sqrt{2}-1$. For any value of y within these bounds, x is uniquely determined by (1) and accompanied by *equalities* $d_1 = d_2$ and $d_3 = d_4$.

We showed above that for every $y \in \left[0.25, \sqrt{2}-1\right]$ there is a 6-coloring of type (1,1,1,1,1, α), but what values can α have? Surely, $\alpha = \dfrac{d_4}{d_2}$. By making a substitution $Y = \sqrt{2-4y}$, where $Y \in \left[0.25, \sqrt{2}-1\right]$, we get

$$\alpha^2 \;=\; \frac{Y^4 - 4Y^3 + 8Y^2 - 8Y + 4}{Y^4 - 8Y^3 + 24Y^2 - 32Y + 20}$$

By substituting $Z = Y - 2$, where $Z \in \left[-\sqrt{2}, -1\right]$, we get:

$$\alpha^2 \;=\; 1 + \frac{4Z(Z^2 + 2Z + 2)}{Z^2 + 4}$$

To observe the behavior of the function α^2, we compute its derivative:

$$(\alpha^2)' \;=\; -\frac{4}{(Z^4+4)^2}(Z^6+4Z^5+6Z^4-12Z^2-16Z-8)$$

We are positively lucky, for this sixth degree polynomial can be decomposed:

$$(\alpha^2)' \;=\; -\frac{4}{(Z^4+4)^2}(Z^2-2)\left[(Z+1)^4+2(Z+1)^2+1\right]$$

Hence, on the segment of our interest $Z \in \left[-\sqrt{2}, -1\right]$, the extremum of α^2 occurs when $Z = -\sqrt{2}$. Going back from Z to Y to y, we see that on the segment $y \in \left[0.25, \sqrt{2}-1\right]$ the function $\alpha = \alpha(y)$ increases from $\alpha = \sqrt{2}-1 = 0.41421356$ (i.e. 6-coloring of problem 13) to $\alpha = \dfrac{1}{\sqrt{5}} = 0.44721360$ (i.e., 6-coloring of problem 10). Since the function $\alpha = \alpha(y)$ is continuous and increasing on $\left[0.25, \sqrt{2}-1\right]$, it takes on *each* intermediate value from the segment $\left[\sqrt{2}-1, \dfrac{1}{\sqrt{5}}\right]$, and only *once.*

For every angle C between the small and the large squares (see figure 18), there are, unique, sizes of the two squares, such that the constructed 6-coloring has type (1,1,1,1,1, α) for a uniquely determined α.

Remark: the problem of finding the 6-realizable set X_6 has a close relationship with the problem of finding the chromatic number of the plane χ. Its solution would shed light if not solve the chromatic number of the plane problem:

if $1 \notin X_6$, then $\chi = 7$
if $1 \in X_6$, then $\chi \le 6$

Open Problem 16. (A. Soifer [39]). Find X_6.

I am sure you understand that this short problem is extremely difficult.

5. The Art of Coloring: Ode to Bees

I have one tiny regret. Hadwiger-Isbell's, Stechkin's and my ornaments (Figures, 9, and 13 respectively) delivered nothing new to the world of *art*. Artists of China, India, Persia, Turkey, and Europe have known our ornaments for over 1,000 years. Figures 21, 22, and 23 reproduced with the kind permission of the Harvard-Yenching Institute from the wonderful 1937 book *A Grammar of Chinese Lattice* by Daniel Sheets Dye ([4]), show how those ornaments were implemented in an old Chinese lattice.

But even our ancestors did not invent the honeycomb (Figure 21). Bees were first. (Three cheers to the bees!).

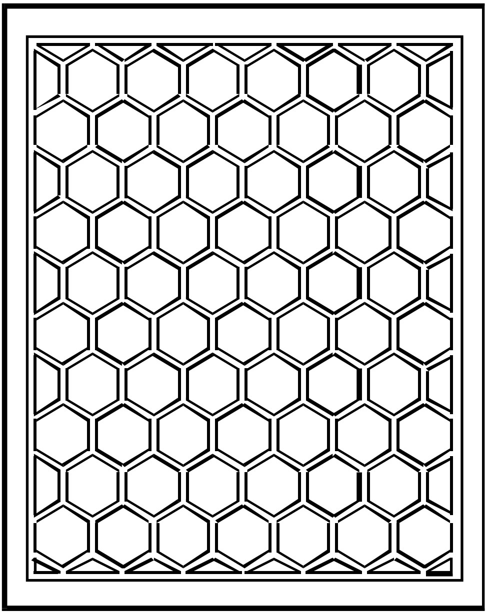

Figure 21

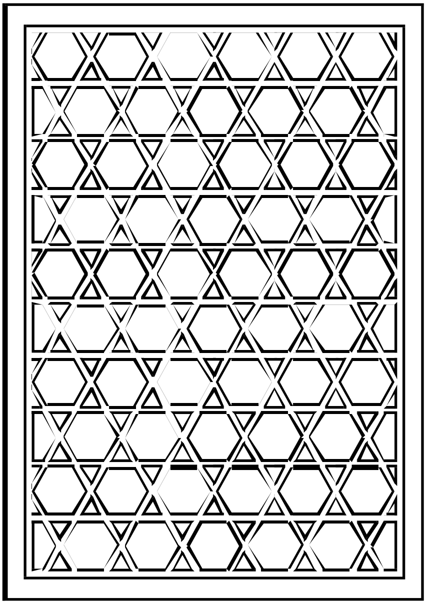

Figure 22

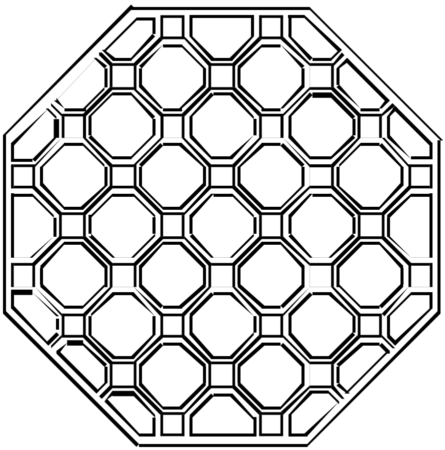

Figure 23

6. Epilogue

'If I live, I hope to preach next year. If I don't, I hope that somebody will lecture in my memory.' These were the last words of Paul Erdös's talk on March 10, 1994 in Boca Raton [14].

On March 13, 2003 we will celebrate the 90th anniversary of Paul Erdös' birthday. Paul created enough great open problems to last for centuries. Chromatic number of the plane was not his problem, but he liked it a lot. And he has single-handedly kept this problem alive. This talk and this paper are dedicated to the memory of Paul.

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Colourings of 3-space using Lattice/Sublattice Schemes

Part 2[1]

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1. The problem.

- What is the smallest number of colours needed to colour every point of n-space so that no two points distance 1 apart have the same colour? (We say 1 is an excluded distance.)
- What colourings use this number of colours?
- (Clearly we can dilate space so any excluded distances suffice.)

2. Colourings – the *Lattice/Sublattice* colouring scheme.

The Scheme:

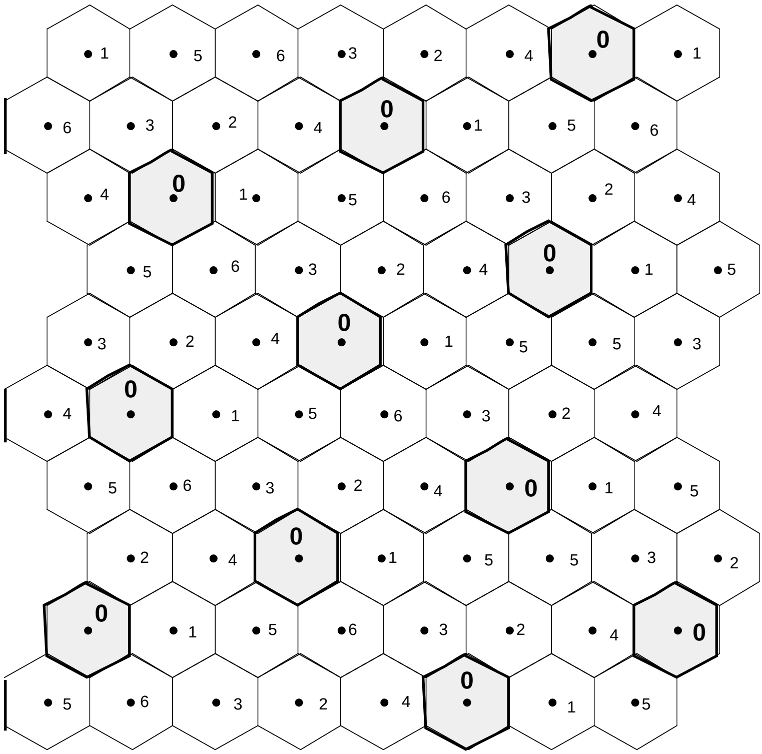
- Take a lattice Λ and use it to tessellate space into nearest neighbour regions (**NNRs**).
- Colour these tiles according to cosets of a sublattice Γ of Λ.

[1]This paper was presented as a keynote address at WFNMC Congress-4, Melbourne August 2002

– So that different tiles of the same colour are at least the diameter of a tile apart.

– Look for a sublattice (a subset of a lattice that is itself a lattice) of low index for an efficient colouring.

For example the 7 colouring of the plane based on the *Hexagonal Lattice* (the black dots) is constructed using NNRs (hexagons) and a hexagonal sublattice of index 7 (black dots labelled 0).



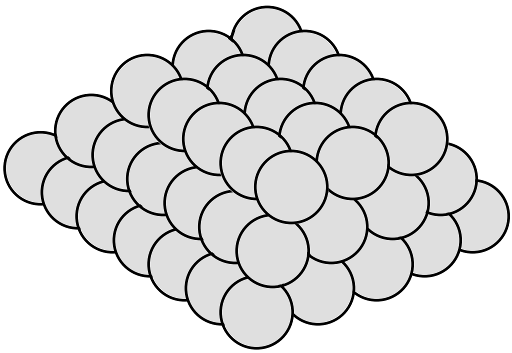
3. In 3-space...

In the early 1990s the most efficient known colouring of 3-space used 21 colours and was based on the *Face Centred Cubic Lattice* **FCC-L** (alternatively known as D_3 or A_3). The FCC-L has *basis* (over the integers) $(1, 1, 0)$, $(1, -1, 0)$, $(1, 0, 1)$ and consists of all vectors with integer coordinates and even parity.

The FCC-L is the most efficient packing lattice in 3 dimensions, meaning that if you had many identical spheres and had to arrange their centres on a lattice the FCC-L is the arrangement that will pack them in as tightly as possible.

In fact Thomas Hales [5] has (purportedly) shown recently that this is the most efficient sphere packing possible regardless of the arrangement of centres (that is, no tighter packing exists) – solving the *Kepler Conjecture.*

It is the packing arrangement often seen at greengrocers, where the horizontal layers are arranged in square grids, and these horizontal layers are stacked so that oranges in neighbouring horizontal layers sit in the gap between 4 oranges.



The FCC-L close packing.

4. The 21 colouring of 3-space.

For the 21-colouring of 3-space, the sublattice Γ of FCC-L used to colour the NNRs has basis (1,2,3), (-3,1,-2), (-1,-3,2) where

$$\begin{array}{rcrrr} (1,2,3) & = & 0(1,1,0) & -2(1,-1,0) & +3(1,0,1), \\ (-3,1,-2) & = & 0(1,1,0) & -1(1,-1,0) & -2(1,0,1), \\ (-1,-3,2) & = & -3(1,1,0) & +0(1,-1,0) & +2(1,0,1) \end{array}$$

giving the index of Γ in FCC-L as

$$\begin{vmatrix} 0 & -2 & 3 \\ 0 & -1 & -2 \\ -3 & 0 & 2 \end{vmatrix} = 21.$$

We require the NNRs that are coloured the same to be greater than the covering diameter $\mathcal{D}\,(=2)$ apart.

NNRs centred on $(0,0,0)$ and any permutation of $(1,2,3)$ (which includes all the shortest non-zero vectors in Γ) are distance $\sqrt{14/3}\,(>2)$ apart. The next shortest non-zero vectors in Γ have length $\sqrt{20}$ and so the NNRs are at least $\sqrt{20} - 2 \times 1 > 2$ apart. So we have an excluded distance a little greater than 2. In fact the range of excluded distances is $(2, \sqrt{\frac{14}{3}}) = (\mathcal{D}, \sqrt{\frac{7}{6}}\mathcal{D}) \approx (\mathcal{D}, 1.08\mathcal{D})$.

5. An 18 colouring of 3-space.

For an 18 colouring of 3-space [3], the sublattice Γ of the BCC-L used to colour the NNRs has basis (8,0,4), (4,8,0), (0,4,8) where

$$\begin{array}{rcrrr} (8,0,4) & = & 1(-2,2,2) & +3(2,-2,2) & +2(2,2,-2), \\ (4,8,0) & = & 2(-2,2,2) & +1(2,-2,2) & +3(2,2,-2), \\ (0,4,8) & = & 3(-2,2,2) & +2(2,-2,2) & +1(2,2,-2). \end{array}$$

giving the index of Γ in BCC-L as

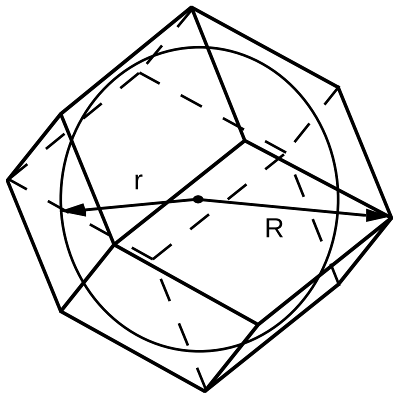
$$\begin{vmatrix} 1 & 3 & 2 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{vmatrix} = 18.$$

Let us check that the NNRs are at least a covering diameter apart. NNRs centred on $(0, 0, 0)$ and the nearest vectors of Γ, for example $(8, 0, 4)$ are exactly a covering diameter apart (as $(8, 0, 4) = 4 \times (2, 0, 1)$ with (2,0,1) being a vertex of the NNR of the BCC-L) and we exclude the distance $\mathcal{D}$ (the covering diameter) from the colouring by colouring diametrically opposite vertices of the NNR different colours. Note that $\mathcal{D}$ is the only excluded distance.

By perturbating the lattice Γ (see section 8) and as a result the tessellating lattice BCC-L, we can find a tessellating lattice Λ with basis $(1, 4, -2), (-2, 1, 4), (4, -2, 1)$ and sublattice Γ with basis $(9, 9, 0), (0, 9, 9)$, $(9, 0, 9)$ (which is isomorphic to $9 \times$ FCC-L) the covering diameter $\mathcal{D}$ is $\sqrt{\frac{113}{3}}$ and NNRs coloured according to Γ are distance $\sqrt{\frac{131}{3}}$ apart. So $(\mathcal{D}, \sqrt{\frac{131}{113}}\mathcal{D}) \approx (\mathcal{D}, 1.077\mathcal{D})$ is a range of excluded distances.

6. Can we do better? — The 15 colouring of 3-space.

If balloons were packed according to the FCC-L and then inflated (uniformly) so that no gaps existed in between them, each would be shaped as the NNR with 12 rhombic faces, shown below.



The NNR of FCC-L has inradius (packing radius) $r = \frac{\sqrt{2}}{2}$ and covering radius $\mathcal{R} = 1$.

The optimal packing properties of FCC-L can be expressed algebraically as the NNR of FCC-L, N_{FFC} say, which satisfies

$$\begin{aligned} \text{Volume}(N_{\mathrm{FFC}})/r^3 &= 4\sqrt{2} \\ &= \min_{P \in SF} \text{Volume}(P)/r^3 \end{aligned}$$

where SF is the set of polyhedra that fill 3-space (under translations) and r is the radius of the largest sphere that can fit inside the polyhedron P.

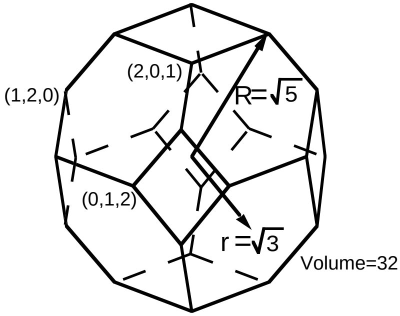
That is the inradius r is as large as possible for a given volume amongst space filling polytopes.

For the colouring problem we really want a tiling polyhedron (NNR) that has a small diameter $\mathcal{D} = 2 \times \mathcal{R}$ for a given volume, if we restrict our polyhedra to be the NNRs of lattices we are essentially asking for the best *covering lattice* (the lattice arrangement of centres of overlapping spheres that cover all of 3-space with minimal total volume).

The best covering lattice is known in 3-space to be the body centred cubic lattice BCC-L [1] which has basis $(-2, 2, 2),\ (2, -2, 2),\ (2, 2, -2)$ and can be thought of as the union of $(4\mathcal{Z})^3$, the cubic lattice (with side length 4) with the centres of these cubes, that is

$$\text{BCC-L} = (4\mathcal{Z})^3 \cup ((2, 2, 2) + (4\mathcal{Z})^3).$$

The lattice is also known as A_3^* and D_3^*. The NNR of the BCC-L is drawn below.

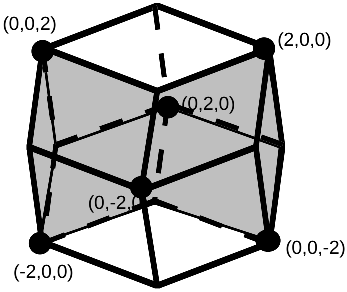


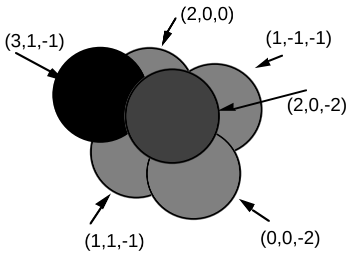
The NNR of the BCC-L.

Using the BCC-L as a tessellating lattice and using *Mathematica*^{TM} (to search among NNRs at least a covering diameter away from the NNR centred at the origin) to find a sublattice Γ of small index, a 15 colouring was easily found. Namely Γ has basis (-2,6,6), (6,-6,2), (2,6,-6) where

$$\begin{array}{rcrrr}
(-2,6,6) & = & 3(-2,2,2) & +1(2,-2,2) & +1(2,2,-2), \\
(6,-6,2) & = & -1(-2,2,2) & +2(2,-2,2) & +0(2,2,-2), \\
(2,6,-6) & = & 0(-2,2,2) & -1(2,-2,2) & +2(2,2,-2).
\end{array}$$

Now BCC-L has $\mathcal{D} = 2\sqrt{5}$ and NNRs at least $\mathcal{D} = 2\sqrt{5}$ apart so by colouring diametrically opposed vertices different colours, we get $\mathcal{D} = 2\sqrt{5}$ as an excluded distance.





The 15-cluster of the 15 colouring consists of 14 NNRs centered on the vertices of the rhombic dodecahedron pictured together with one NNR at the centre of the rhombic dodecahedron.
The 18-cluster of the 18 colouring contains 2 extra NNRs for each of the 6 shaded faces of the rhombic dodecahedron. These extra NNRs contact the faces as shown in the right of the illustration. The medium shaded NNR is shared between 3 18-clusters and the dark shaded NNR is shared between 6 18-clusters (and is in fact a deep hole of the sublattice Γ), this gives us 15+6/3+6/6=18 NNRs in the 18-cluster.

7. Improving the 15 colouring of 3-space.

We may however improve on this so that $\{\mathcal{D}\}$ is not the only excluded distance but we have a range $[\mathcal{D}, \mathcal{D}+\epsilon)$ of excluded distances.

The idea we employ is that of perturbing the basis vectors

$$\{b_1, b_2, b_3\} = \{(-2, 2, 2),\ (2, -2, 2),\ (2, 2, -2)\}$$

of BCC-L, the tessellating lattice and implicitly perturbing the basis vectors of Γ, $\{l_1, l_2, l_3\}$.

$$\begin{tikzcd}
\Gamma \arrow[r, "P\ \text{(perturbation)}"] & \Gamma' \cong D_3 \\
D_3^* \arrow[u, "T(D_3^*)"'] \arrow[r, "P\ \text{(perturbation)}"] & \Lambda \arrow[u, "T(\Lambda)"']
\end{tikzcd}$$

A commutative diagram illustrating the linear perturbation P improving the 18-colouring. (The map T is a bijective linear map from the underlying lattice to the sublattice.)

So we have $l_1 = 3b_1 + 1b_2 + 1b_3, l_2 = -1b_1 + 2b_2 + 0b_3, l_3 = 0b_1 - 1b_2 + 2b_3$ and leave the perturbation of BCC-L as a reasonable covering lattice and making the perturbation of the sublattice Γ closer to FCC-L so that it has better packing properties.

We thus obtain a tessellating lattice Λ with basis $(2,3,0),(-2,0,3)$, $(2,-3,0)$ which has NNRs with a covering diameter of $\mathcal{D} = \sqrt{22} \approx 4.69042$ and the colouring sublattice Γ with basis

$$\begin{array}{rcllll}
(6,6,3) & = & 3(2,3,0) & +1(-2,0,3) & +1(2,-3,0), \\
(-6,-3,6) & = & -1(2,3,0) & +2(-2,0,3) & +0(2,-3,0), \\
(6,-6,-3) & = & 0(2,3,0) & -1(-2,0,3) & +2(2,-3,0).
\end{array}$$

With this sublattice colouring, distinct NNRs are distance $\sqrt{\frac{389}{17}} \approx 4.78355$ apart, which means that $(\mathcal{D}, \sqrt{\frac{389}{374}}\mathcal{D}) \approx (\mathcal{D}, 1.020\mathcal{D})$ is the range of excluded distances [4].

In fact the 15 colouring with the largest excluded distance range $\approx (\mathcal{D}, 1.027\mathcal{D})$ is based on perturbing BCC-L to the lattice Λ with a basis consisting of unit vectors which have dot products $-\alpha, -\alpha$, $2\alpha - 1$ with each other where $\alpha \approx 0.3137$ and is a root of the polynomial $3 - 10x - 3x^2 + 14x^3$. The exact excluded distance range is

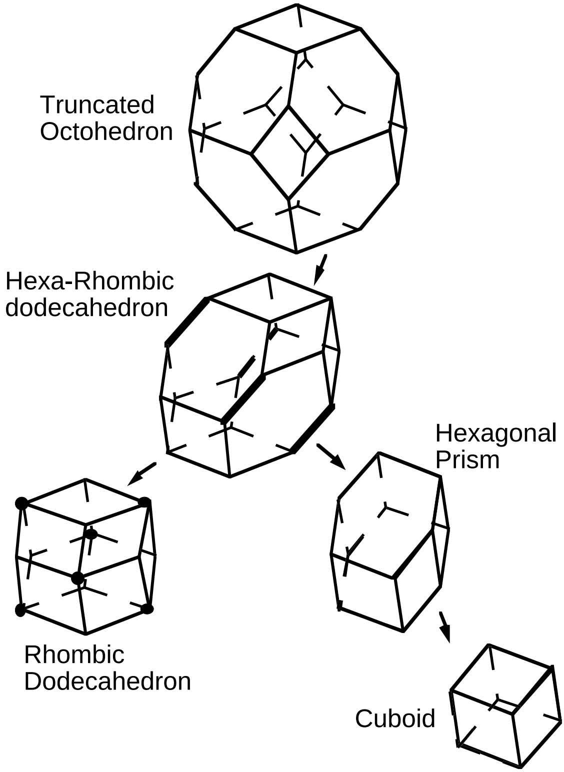
$$(\mathcal{D}, \sqrt{\frac{1+3\alpha+4\alpha^2}{2+\alpha-\alpha^2}}\mathcal{D}) \approx (\mathcal{D}, 1.027\mathcal{D}).$$

8. Using lattice/sublattice colouring schemes a 15 colouring of 3-space is the best possible.

The 15-colouring was found so easily using the computational tools developed it seemed natural to try to find an even more efficient colouring.

This proved to be hard, even when tricks were employed such as leaving vectors in BCC-L that were too close (to have NNRs at least $\mathcal{D}$ apart), hoping that the perturbation process would make them far enough apart.

Eventually it was seen that this was not possible – in broad terms because a generic 3-space NNR has 14 faces.



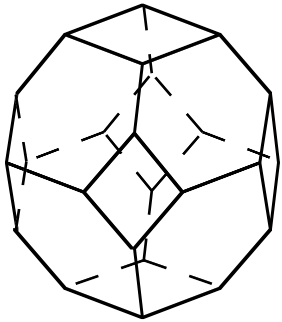
The 5 combinatorially distinct types of NNR in 3-space (Fedorev 1885, 1891).

Theorem 8.1 *In 3-space any lattice/sublattice colouring scheme requires at least 15 colours to have $\mathcal{D}$ (the covering diameter of the NNR of the tessellating lattice) as an excluded distance.*

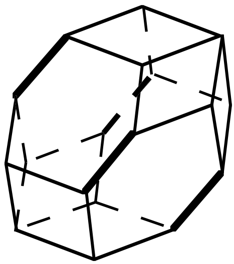
Proof: The key idea is that any two NNRs contacting a common NNR are $< \mathcal{D}$ apart. (Unless they contact this common NNR at diametrically opposite vertices distance $\mathcal{D}$ apart.)

In this proof we look at each of the 5 combinatorial types of NNR and show that each has at least 14 NNRs surrounding a central NNR that are less than $\mathcal{D}$ apart.

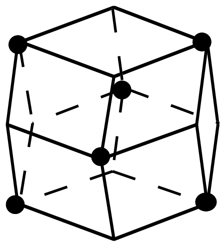
(This may also be done for all NNRs simultaneously by looking at minimal vectors in $\Lambda \bmod (2\Lambda)$.) [4]



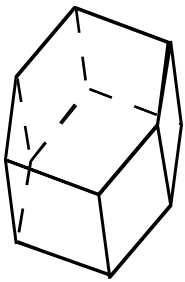
1. Consider tessellating 3-space with a *generic* NNR - *truncated octahedral NNR* (the NNR of BCC-L is an example):
14 other NNRs contact this on faces and clearly all 1+14+0+0=15 NNRs must be coloured differently to have $\mathcal{D}$ as an excluded distance.



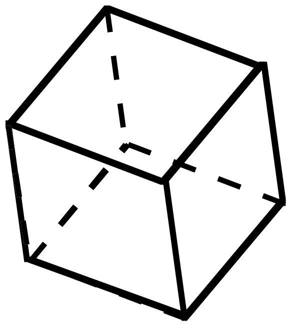
2. Consider tessellating 3-space with a *hexa-rhomboid* NNR:
12 other NNRs contact this on faces and 4 other NNRs contact this on edges (those separating the hexagonal faces), these 1+12+4+0=17 NNRs must be coloured differently to have $\mathcal{D}$ as an excluded distance.



3. Consider tessellating 3-space with a *rhombic dodecahedral* (the NNR of FCC-L is an example):
12 other NNRs contact this on faces and 3 pairs of NNRs contact this NNR on diametrically opposite vertices (those where 4 faces meet), each NNR in a pair may be coloured the same (as they are $\mathcal{D}$ apart) but the 3 pairs must be coloured differently, thus at least 1+12+0+3=16 colours are required for $\mathcal{D}$ to be an excluded distance.



4. Consider tessellating 3-space with a *hexagonal prismic* NNR:
8 other NNRs contact this on faces and 12 other NNRs contact this NNR on edges (those that bound the hexagonal faces), thus all 1+8+12+0=21 NNRs need to be coloured differently for $\mathcal{D}$ to be an excluded distance.



5. Consider tessellating 3-space with a *cuboidal* NNR:
6 other NNRs contact this on faces, 12 others on edges and 4 pairs of NNRs contact this NNR on diametrically opposite vertices, thus at least 1+6+12+4=23 colours are required for $\mathcal{D}$ to be an excluded distance. □

We have thus shown that lattice/sublattice colourings with $\mathcal{D}$ as an excluded distance need at least 15 colours [4].

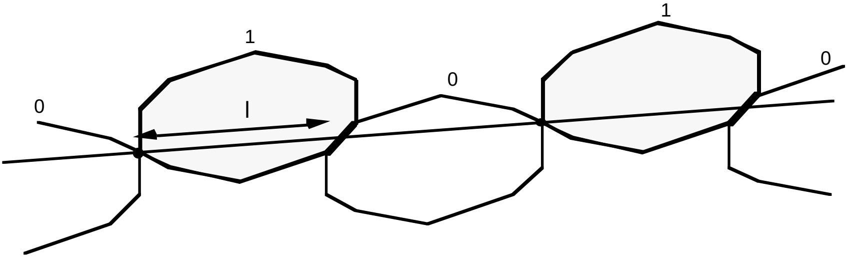
We now show that for any excluded distance 15 colours are necessary.

Let N be the NNR of Λ centred on the origin.

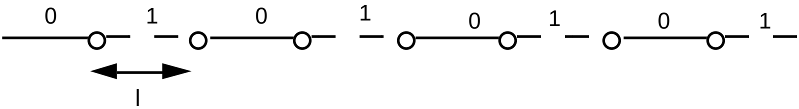
Theorem 8.2 *If 3-space is coloured using a lattice/sublattice colouring scheme then the colouring requires at least* 15 *colours to exclude any distance.*

Proof. Suppose to the contrary we colour R^3 using a sublattice Γ of index < 15 in the tesselating lattice Λ, then by Theorem 8.1 there are two NNRs $\lambda_1 + N$ and $\lambda_2 + N$ that are necessarily coloured the same (colour 0) and make contact with N (coloured with colour 1 say) at facets that are not diametrically opposite vertices of N.

The situation is illustrated below using decagons (decagons cannot be NNRs for any lattice but this shape illustrates the argument better than actual NNRs.)



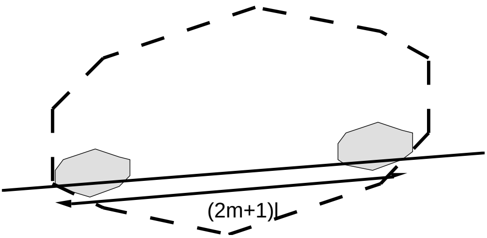
Consider the straight line that passes through the centroids of the contact faces between $N, \lambda_1 + N$ and $\lambda_2 + N$. Due to the translational symmetry of lattices this line is only interior to NNRs coloured 0 and 1. Considering only this line we have the situation as shown in the diagram below.



Let l be the length of the line in any one Voronoi region. Note that the points corresponding to the circles may have colours different from 0 and 1 (if the contact faces are of dimension less than 2).

Clearly the only candidates for excluded distance are odd multiples of l.

Let us contract R^3 by a factor of $(2m+1)$, that is map x to $x/(2m+1)$. The situation for a segment of the line of length $(2m+1)l$ is shown in the diagram below.



The endpoints will locally look as they do on the boundary of N. Clearly the line segment can be both shortened infinitesimally (with both endpoints the same colour) and can be lengthened infinitesimally (with both endpoints the same colour) and so by continuity the length of the line segment can remain the same (with both endpoints the same colour).

Thus $(2m + 1)l$ cannot be an excluded distance if Γ has index < 15 in Λ. From this it follows that if we colour R^3 using a lattice/sublattice colouring scheme with less than 15 colours there can be no excluded distance. $\square$

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Math Fairs[1]

Andy Liu

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Basic Principles

A Math Fair consists of a number of booths in a carnival setting, each booth presenting an interactive mathematical puzzle or game. The primary aim is to promote interest in mathematics, and develop problem-solving skills.

A typical Math Fair booth consists of a problem presented in very brief verbal description, but with manipulative pieces which ideally should help people solve the problem, in addition to making it come alive. The problem does not have to be elaborate, or new, since most will be new to the students themselves. They are not provided with the solutions, and must work it out for themselves. This way, they will be well-prepared to face the audience.

It should be pointed out that a booth at a Math Fair is not meant to be a mini-station for teaching concepts. Also, problems which are presented in poster form and require pencil-and-paper work on the side are not suitable for Math Fairs.

[1]This is an enhanced version of the text of a plenary speech delivered at the Fourth Congress of the World Federation of National Mathematics Competitions in Melbourne, Australia on August 8, 2002.

The first Math Fair in Edmonton was held sometime last century at Our Lady of Victories Catholic Elementary School, under the direction of Vice-Principal Mike Dumanski. There was a dress rehearsal in the school gymnasium, and a formal presentation in a local shopping mall two days later. It was an inclusive event in that every child in the school participated. They were divided into groups of three or four by grade levels, with each group responsible for a booth.

A most important feature of the Math Fair is that there is no judging, and no awarding of prizes. Children learn to do things for their intrinsic value. They are genuinely interested in what their schoolmates have been working on, and try out the problems on one another. It is doubtful if this congenial atmosphere is compatible with a more competitive environment.

My colleague Ted Lewis has taken the Math Fair one step further and makes it a compulsory part of MATH 160, a course for students in Elementary Education at the University of Alberta. The university students prepare a Math Fair first, presenting it in two sessions so that they can assess the work of one another. Then they take the show on the road and set up their Math Fair in the gymnasium of an elementary school for their students, who come in one class at a time. Using this as a model, the elementary school students produce their own Math Fair.

In this century, the waiting list of schools waiting for the Math Fair has grown so long that instead of taking the Math Fair to them, Ted sets up the Math Fair on university campus and brings in children from several schools at a time. This has become a very popular event.

Ted Lewis has written a Math Fair Booklet which the University of Alberta and the Pacific Institute of Mathematical Sciences sell at nominal cost. Those who are interested can contact Ted at tlewis@math.ualberta.ca.

Borrowing Problems

The most often asked question about the Math Fair is: Where do you find the problems? Some of the time, I make them up. Most of the time, I borrow them from elsewhere. The best sources are the writings of Martin Gardner, and the vast Russian literature. As will be seen, they also borrow from each other.

Some problems are ready-made for the Math Fair. This well-known example is taken from [4], titled “Wolf, Goat, and Cabbage”. It is later discussed in [1] within a broader theoretical context.

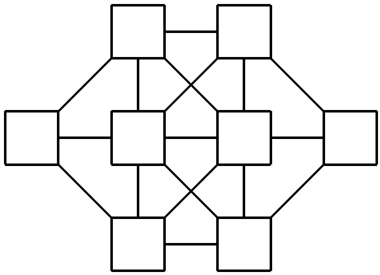
A man has to take a wolf, a goat, and some cabbage across a river. His rowboat has enough room for the man plus either the wolf or the goat or the cabbage. If he takes the cabbage with him, the wolf will eat the goat. If he takes the wolf, the goat will eat the cabbage. Only when the man is present are the goat and the cabbage safe from their enemies. All the same, the man carries wolf, goat, and cabbage across the river. How?

This is not a hard problem. After solving it, we have to figure out how to present it. At the inaugural Math Fair, an actual basin of water was used to represent the river. Unfortunately, the boat was so small that whenever any of the figures used to represent the man, the wolf, the goat or the cabbage was put into it, it sank! A larger boat was used when the Math Fair moved to the mall, but now there was plenty of room for all four of them. Also, by the end of the evening, there was a real river under the table, prompting a teacher to exclaim in exasperation, “No more water next time!”.

Perhaps a virtual river drawn on paper will do. However, having the figures to move around not only makes the problem come alive, it actually helps us visualize what moves are legal, and what moves are sensible.

Even if a problem is ready-made for the Math Fair, there may still be room for improvement. Consider this example from [3], titled “Digit-Placing”, which appears later in [7].

The digits from 1 to 8 are to be placed in the eight boxes in the diagram below, with this provisio: no two consecutive digits may go in boxes that are directly connected by a line.



Again, this problem is not hard. However, as phrased, it is an all-or-nothing kind of problem — either you can solve it or you cannot. I suggest the following modification. Place the eight digits any way you wish. You score 1 point for each pair of boxes connectd by a line and containing two digits which are not consecutive. Thus your score is 17 points if you can solve the original problem. However, even if you do not succeed in solving it completely, you can still obtain a positive score.

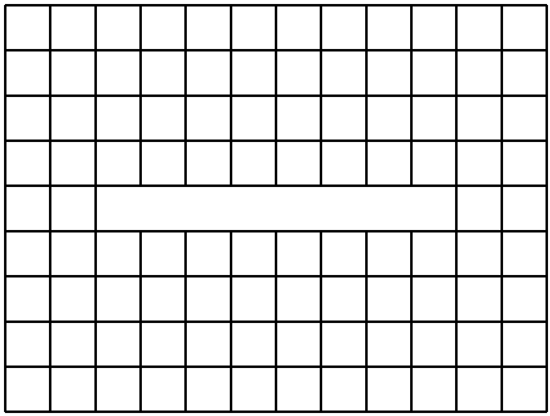
Not every problem is suitable for the Math Fair format. Here is an example from [9], titled “Family Planning”.

A family of four (father, mother, son and daughter) went on a hike. They walked all day long and when evening was already drawing on, came to an old bridge over a deep gully. It was very dark and they had only one lantern with them. The bridge was so narrow that it could hold no more than two persons at a time. Suppose it takes the son 1 minute to cross the bridge, the daughter 3 minutes, the father 8 minutes, and the mother 10 minutes. Can the entire family cross the bridge in 20 minutes? If so, how? (When any two persons cross the bridge, their speed is equal to the slower one. Also, the lantern must be used while crossing the bridge.)

Despite the superficial resemblance to the “Wolf, Goat, and Cabbage” problem, it is not easy to find a meaningful way to represent the relative speeds of the family members. Although this is an excellent problem, I have to reluctantly abandon it as far as the Math Fair is concerned.

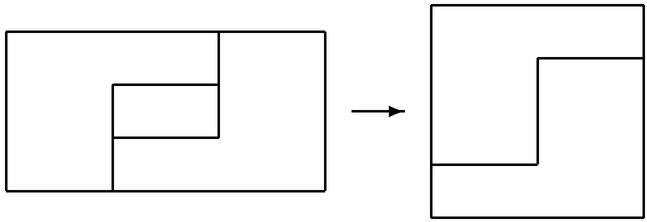
There are also problems which at first sight are totally unsuitable for the Math Fair. Nevertheless, with some modificiation, they can be used. One of the best example is from [5], which appears later in [2] under the title “The Damaged Patchwork Quilt”.

Part of the centre of the 9×12 *patchwork quilt in the diagram below became worn, making it necessary to remove 8 of the 108 squares as indicated. Cut the quilt into just two parts that can be sewn together to make a* 10×10 *quilt.*

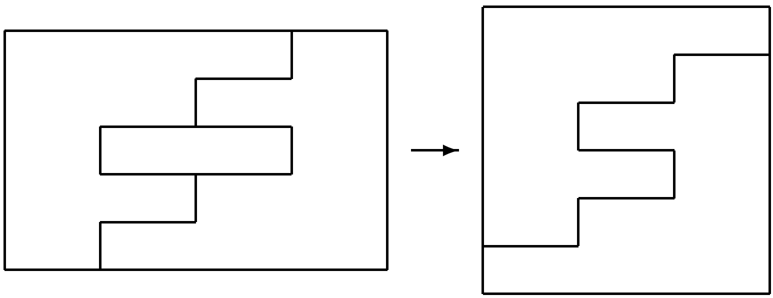


This is a difficult problem. One approach is to “downsize” it, which also yields the benefit of providing subsidiary problems at lower levels. We may generalize the 9×12 rectangle with a 1×8 hole in the middle to a $2n - 1 \times 2n + 2$ rectangle with a $1 \times 2n - 2$ hole in the middle, and the 10×10 square to an $2n \times 2n$ square. The original problem is the case $n = 5$.

For $n = 1$, we have a 1×4 rectangle with a 1×0 hole in the middle and a 2×2 square. This is going too far down. The case $n = 2$ turns out to be quite easy. Since the top row of the rectangle has length 6 while the target square has side 4, we must have a cut as indicated in the diagram below. Another cut in the symmetric position yields the desired two parts.



In the case $n = 3$, we also have to make a cut along the top edge of the 5×8 rectangle at the point 6 units from a corner. However, if we cut through to the hole as we did in the case $n = 2$, it will not work. After a bit of fiddling around, we may arrive at the following solution.



From these preliminary investigations, it is not unreasonable to assume that the two parts in the solution are congruent to each other, and placed rotationally symmetric to each other. However, we are certainly not confident at this point about solving the original problem.

Perhaps we should pause and consider the companion dilemma of how to present it if we somehow suceeed in finding a solution. The obvious model is providing lots of scissors and pre-cut rectangles with holes, but this leaves a lot to be desired. Mixing kids and scissors is never a good idea. Besides, the floor will be littered.

The most important drawback is that this model does not help us find a solution to the problem. Cutting at random will not work. In fact, we have to have the solution already in our mind before we can do any meaningful cutting. This problem was offered and rightly rejected at the inaugural Math Fair. It was years later that I finally came up with a fantastic model. Let me illustrate with the case $n = 4$.

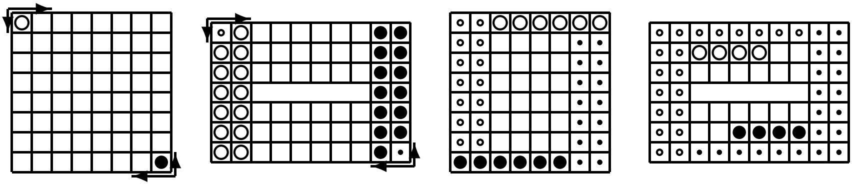
Draw two playing boards, one a 7×10 rectangle with a 1×6 hole in the middle, and the other an 8×8 square. Provide a large supply of bingo chips of two different colours, say yellow and blue. Our objective is to fill both boards with chips in such a way that those of the same colour form the same shape on both boards. This will yield the solution to the problem.

We begin by placing a yellow chip at the top left corner of the square board and a blue chip at the bottom right corner. This signifies that these two squares do not belong to the same part, a most reasonable assumption. We define these two corners as the principal corners of the respective parts. We mark them in the same orientation since we anticipate that the two parts are in rotational symmetry.

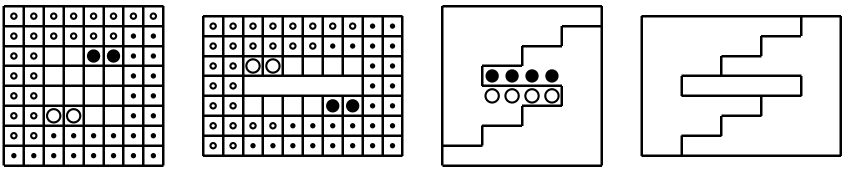
We now move over to the punctured rectangular board and mark the principal corners as indicated. Besides playing the chips at the corners, we observe that in the square board, the yellow chips cannot extend beyond the eighth column from the left while the blue chips cannot extend beyond the eight column from the right. This allows us to place the additional chips on the rectangular board.

Moving back to the square board, we can fill the top row with yellow chips and the bottom row with blue ones. This is because in the rectangular board, the yellow chips cannot extend beyond the seventh row from the top and the blue chips cannot extend beyond the seventh row from the bottom.

Moving over to the rectangular board, we notice that the two blue chips on the seventh row from the bottom cannot extend at all to the left. This means that we must have six yellow chips on the second row from the top, going from left to right. Similarly, there must be six blue chips on the second row from the bottom, going from right to left.



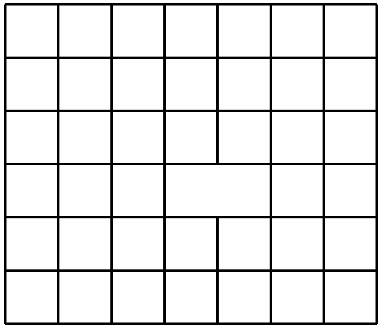
We have now arrived at the position in the diagram above. Continuing with the same strategy, we can complete the solution of the problem as in the diagram below.



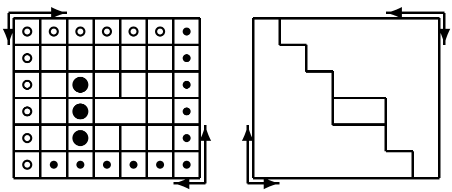
Not only have we found a good way of presenting the problem, we have actually found an excellent way of solving it. This is serendipity!

This approach may be used to solve related problems. In the above problem, the two parts we end up with are congruent to each other. Of course, they also have to fit together to form a square. Since the punctured rectangle has rotational symmetry, merely asking for a division into two congruent parts is pointless. However, if a figure does not have any obvious symmetry, the task may be quite challenging. Here is an example from [6], titled "Board".

Divide the board in the diagram below into two congruent parts.



There are two choices for the principal corners. Since the board does not have rotational symmetry, we choose opposite orientations for the corners. In the first attempt as shown in the left diagram below, the boundary squares are easily filled out. We observe that in the third row from the bottom, we can have at most two blue chips going from right to left. This means that in the third column from the left, we must have at least four blue chips going from bottom to top. This in turn means that in the second column from the bottom, we must have at least four yellow chips going from left to right. However, this is impossible as the blue chips just placed are in the way. The other choice leads easily to the solution shown in the right diagram below.



It has been said that it is a trick if you use it once, and a method if you use it more than once. We have discovered a good method for solving a certain class of problems.

Exercises

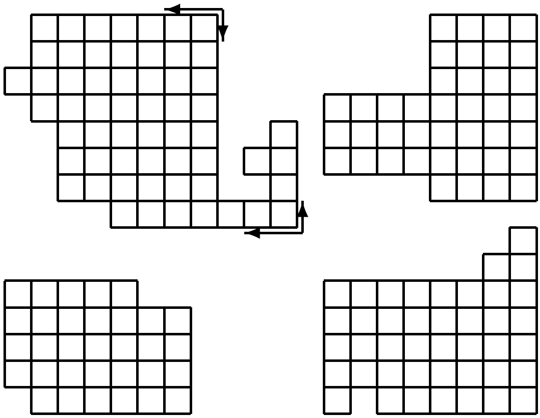
1. How can a goat, a head of cabbage, two wolves, and a dog be transported across a river if it is known that the wolf is "culinarily partial to" goat and dog, the dog is "on bad terms with" the goat, and the goat is "not indifferent" to the cabbage? There are only three seats in your boat, so you can only take two passengers — animal or vegetable — at a time. [8]

2. Make the "Digit-Placing" problem multi-levelled by devising a diagram with n boxes into which the digits 1 to n are to be placed under the same proviso, for $4 \leq n \leq 7$, such that the solution is essentially unique.

3. Solve the "Family Planning" problem.

4. Solve the "Damaged Patchwork Quilt" problem.

5. Dissect each of the following diagrams into two congruent pieces. As a hint, the principal corners of one of them is given. [6]



References

[1] Martin Gardner, '*The Last Recreations*', Springer-Verlag, New York, (1990), pages 114–115.

[2] Martin Gardner, *The Last Recreations*, Springer-Verlag, New York, (1990), pages 249–250.

[3] Martin Gardner, *The Unexpected Hanging and other Mathematical Diversions*, University of Chicago Press, Chicago (1969/1991), pages76–77.

[4] Boris Kordemsky, *The Moscow Puzzles*, Dover Publications, New York, (1972/1987), pages 4–5.

[5] Boris Kordemsky, *The Moscow Puzzles*, Dover Publications, New York, (1972/1987), pages 64–65.

[6] Rodolfo Kurchan, *Mesmerizing Math Puzzles*, Sterling Publishing Company, New York, (2001) pages 36–38.

[7] *Quantum*, Vol. **0**, #1, January 1990, page 41.

[8] *Quantum*, Vol. **0**, #2, May 1990, page 13.

[9] *Quantum*, Vol. **7**, #5, May/June 1997, page 13.

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WFNMC International & National Awards

David Hilbert International Award

The David Hilbert International Award was established to recognise contributions of mathematicians who have played a significant role over a number of years in the development of mathematical challenges at the international level which have been a stimulus for mathematical learning.

Each recipient of the award is selected by the Executive and Advisory Committee of the World Federation of National Mathematics Competitions on the recommendations of the WFNMC Awards Sub-committee.

Past recipients have been: Arthur Engel (Germany), Edward Barbeau (Canada), Graham Pollard (Australia), Martin Gardner (USA), Murray Klamkin (Canada), Marcin Kuczma (Poland), Maria de Losada (Colombia), Peter O'Halloran (Australia) and Andy Liu (Canada).

Paul Erdös National Award

The Paul Erdös National Award was established to recognise contributions of mathematicians who have played a significant role over a number of years in the development of mathematical challenges at the national level and which have been a stimulus for the enrichment of mathematics learning.

Each recipient of the award is selected by the Executive and Advisory Committee of the World Federation of National Mathematics Competitions on the recommendations of the WFNMC Awards Sub-committee.

Past recipients have been: Luis Davidson (Cuba), Nikolay Konstantinov (Russia), John Webb (South Africa), Walter Mientka (USA), Ronald Dunkley (Canada), Peter Taylor (Australia), Sanjmyatav Urjintseren (Mongolia), Qiu Zonghu (China), Jordan Tabov (Bulgaria), George Berzsenyi (USA), Tony Gardiner (UK), Derek Holton (New Zealand), Wolfgang Engel (Germany), Agnis Andžāns (Latvia), Mark Saul (USA), Francisco Bellot Rosado (Spain), János Surányi (Hungary), Istvan

Reiman (Hungary), Bogoljub Marinkovich (Yugoslavia), Harold Reiter (USA) and Wen-Hsien Sun (Taiwan).

The general meeting of the WFNMC in Melbourne agreed, from 2003, to merge the above two awards into one award titled *the Paul Erdös Award.*

Requirements for Nominations for the Paul Erdös Award

The following documents and additional information must be writtten in English:

- A one or two page statement which includes the achievements of the nominee and a description of the contribution by the candidate which reflects the objectives of the WFNMC.
- Candidate's present home and business address and telephone/telefax number.

Nominating Authorities

The aspirant to the Awards may be proposed through the following authorities:

- The President of the World Federation of National Mathematics Competitions.
- Members of the World Federation of National Mathematics Competitions Executive Committee or Regional Representatives.

The Federation encourages the submission of such nominations from Directors or Presidents of Institutes and Organisations, from Chancellors or Presidents of Colleges and Universities, and others.

* * *

Pigeonholes and Two-way Counting[1]

Anne Penfold Street

Anne Penfold Street is Professor of Mathematics at the University of Queensland in Brisbane, Australia. She gained her Ph D at the University of Illinois.
She has been on the Problems Committee of the Australian Mathematics Competition, the Australian Mathematical Olympiad Committee and the Board of the Australian Mathematics Trust. She has been involved with enrichment programs for secondary students. Her research interests are mostly in combinatorics.

In this paper, we look at two techniques which are useful in solving problems. Although the underlying idea is very simple in both cases, they often allow us to solve quite complicated looking problems; see also [1],[11].

First of all, we state the **Pigeonhole Principle** in several of its different forms.

Suppose that N objects are placed in k pigeonholes. Then:

- if $N > k$, some pigeonhole contains more than one object;

[1]This paper was presented as a keynote address at WFNMC Congress-4, Melbourne August 2002

- if $N > mk$, some pigeonhole contains more than m objects.

If the average number of objects per pigeonhole is a, then:

- some pigeonhole contains at least a objects;
- some pigeonhole contains at most a objects.

Next, we consider a few examples where this idea is applied.

Example 1: **Birthdays**
Among 367 *people, at least two share a birthday.*

In this case, the pigeonholes are the 366 possible dates in a year, where we are allowing for a leap year.

Note here what the pigeonhole principle *doesn't* tell us: we have no idea which two people share a birthday, nor what day the shared birthday might be, nor whether several days of the year are shared birthdays among these particular people, nor even whether all 367 of them were born on Leap Day.

Example 2: **Choosing numbers**
Suppose we choose some 19 *of the* 34 *numbers*

```
1, 4, 7, 10, 13, 16, ..., 97, 100.
```

Then, among our chosen numbers, there are two which sum to 104.

To see this, look at the following table.

The number 103 does not belong to our given set, so the number 1 is not part of a pair that sum to 104. Next we have 16 pairs of numbers such that each pair sums to 104. Finally we have the number 52, again not

	1	
4		100
7		97
10		94
13		91
16		88
19		85
22		82
25		79
28		76
31		73
34		70
37		67
40		64
43		61
46		58
49		55
	52	

one of a pair in this set, summing to 104. These give us 18 pigeonholes: two single numbers and 16 pairs.

Thus we could choose one number from each pigeonhole, a total of 18 numbers, without having any two that sum to 104. But to choose 19 numbers from this set, we must take both numbers from at least one of the 16 pairs. See [3].

Example 3: **Aspirins**

A man takes at least one aspirin a day for 30 *days. If he takes* 45 *aspirins altogether, then in some sequence of consecutive days, he takes exactly* 14 *aspirins.*

Let a_i be the number of aspirins he takes in the first i days. Since he takes at least one aspirin a day, we know that $a_i < a_{i+1}$ for $i = 1, \cdots, 29$.

Thus

$$1 \leq a_1 < a_2 < a_3 < \cdots < a_{30} = 45.$$

When does he take exactly 14 aspirins? We also know that

$$15 \leq a_1 + 14 < a_2 + 14 < \cdots < a_{30} + 14 = 59.$$

Then the 60 numbers $\{a_i | i = 1, \cdots, 30\}$ and $\{a_i + 14 | i = 1, \cdots, 30\}$ take at most 59 distinct values. By the pigeonhole principle, some two of these numbers must be equal.
Since the inequalities given above are all strict, we cannot have $a_i = a_j$ for distinct values of i and j, so for some i and j, we have $a_i = a_j + 14$, and he takes exactly 14 aspirins on days $j + 1, \cdots, i$.

Example 4: **Decimal representation**
In the decimal representation of the number $n = 5 \times 7^{34}$, *some digit must occur at least four times.*

By the Pigeonhole Principle, every integer with more than 30 digits must have at least one of the 10 decimal digits occurring at least four times in its decimal representation.

But $\log n = \log 5 + 34.\log 7 = 29.4324 < 30$. So n has 30 digits.
If none of the 10 decimal digits occurs more than three times, then each of them must occur exactly three times. Then the sum of the digits must be divisible by 3, so that n itself would be divisible by 3.
This is not the case, since the only primes dividing n are 5 and 7. So some digit occurs at least four times. See [6].

Example 5: **Seven real numbers**
Among any seven real numbers $y_1, y_2, \ldots, y_7$, *there are two, say* y_i *and* y_j, *such that*

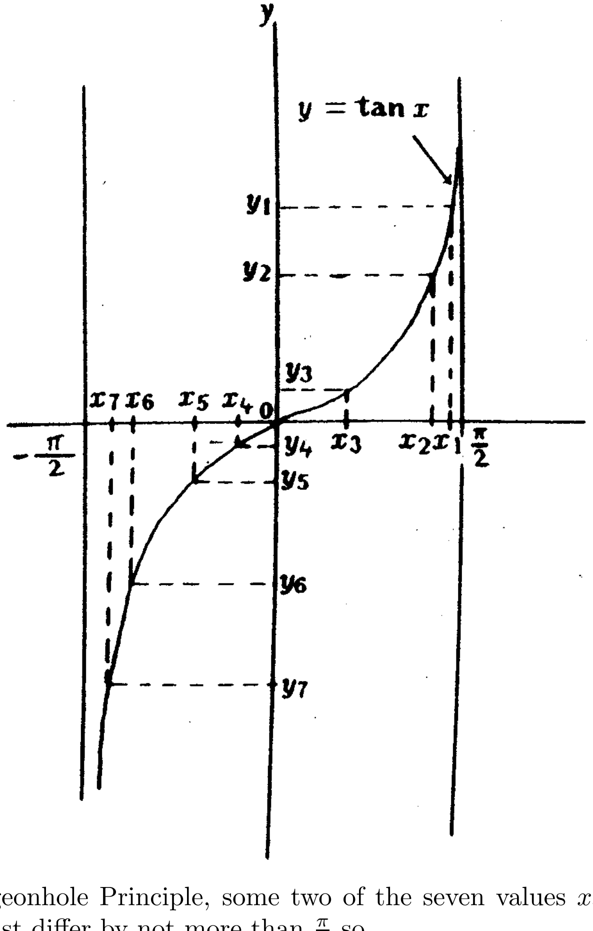
$$0 \leq \frac{y_i - y_j}{1 + y_i y_j} \leq \frac{1}{\sqrt{3}}.$$

This expression brings to mind the formula

$$\tan(x_i - x_j) = \frac{\tan x_i - \tan x_j}{1 + \tan x_i . \tan x_j},$$

especially since $\tan 0 = 0$ and $\tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$.

For $n = 1, 2, \cdots, 7$, let's try $y_n = \tan x_n$, mapping the seven real numbers to seven images in the range between $-\frac{\pi}{2}$ and $+\frac{\pi}{2}$, as shown in the diagram.



By the Pigeonhole Principle, some two of the seven values x_i, for $i = 1, \cdots, 7$ must differ by not more than $\frac{\pi}{6}$ so

$$0 \leq x_i - x_j \leq \frac{\pi}{6}.$$

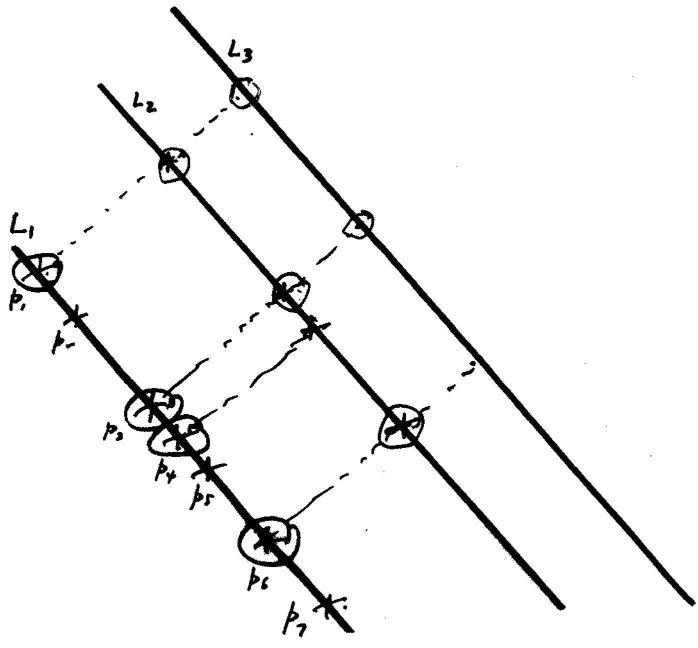
In this range, the tangent is strictly increasing, so we have exactly what we need, namely,

$$\tan 0 \leq \tan(x_i - x_j) \leq \tan\frac{\pi}{6}.$$

See [8].

Example 6: **Coloring the plane**
If we colour the plane in just two colours, Red and Blue, then somewhere in the plane, there must be a rectangle with all four of its corners the same colour.

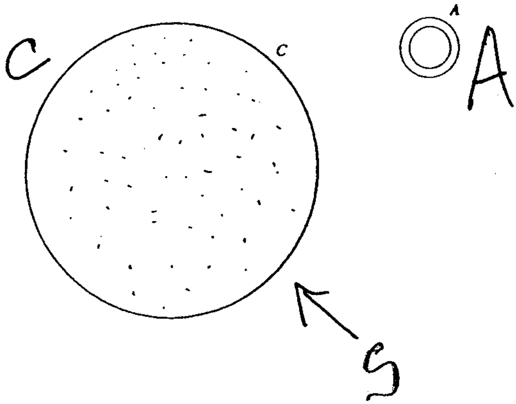
To see this, choose any three parallel lines, L_1, L_2, L_3, in the plane, and any seven points on one of these three lines, say, $p_1, \cdots, p_7$ on line L_1.. Colour each of these seven points either red or blue. By the Pigeonhole Principle, at least four of the points must be the same colour, say, p_1, p_2, p_3, p_4 are red.



Now drop perpendiculars from these four red points to the line L_2, and consider the points q_1, q_2, q_3, q_4 where the perpendiculars from L_1 intersect L_2. If we colour any two of these four points red, say, q_i and q_j, then we have a rectangle with all its four corners red, namely, p_i, q_i, q_j, p_j. To avoid this happening, we must colour at least three of the chosen points on the line L_2 blue, say, q_1, q_2, q_3 are blue.

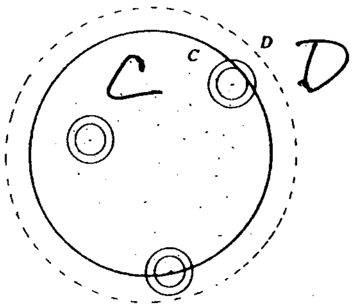
Finally we drop perpendiculars from these points to the line L_3, and consider the points r_1, r_2, r_3 where the perpendiculars from L_2 (and hence also from L_1) meet L_3. If two of these points, say, r_i, r_j, are colored blue, then we have a rectangle q_i, r_i, r_j, q_j with all four of its corners blue. If at most one point is colored blue, then two points, say, r_m, r_n, are colored red, and we have a rectangle $p_m.r_m, r_n, p_n$, with all four of its corners red.

Example 7: **Circle and Annulus**
Suppose that C is a circle of radius 16 units, and that A is an annulus with inner and outer radii 2 and 3 units respectively. Let a set S of 650 points be chosen inside C. Then, no matter how these points are scattered over the circle C, the annulus A can be placed so that it covers at least 10 of the points of S.



Now suppose that a copy of A is centered at each of the 650 points of S.

At a point near the edge, A will stick out past the circumference of C. But since the centre of A is inside C, a circle D, concentric with C and of radius 19 units, will contain all 650 copies of A in its interior.



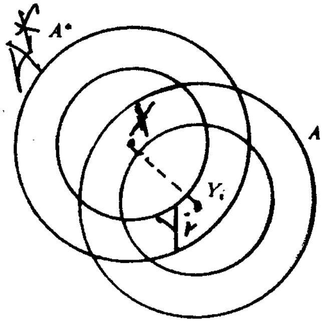
The area of A is $\pi \times 3^2 - \pi \times 2^2 = 5\pi$. Hence 650 copies of A must blanket D with a total coverage of

$$650 \times 5\pi = 3250\pi.$$

If each point of D is covered by at most 9 copies of A, then the total area covering D is at most 9 times the area of D, that is,

$$9(\pi \times 19^2) = 9(361\pi) = 3249\pi.$$

Thus there must be some point X of D on which at least 10 copies of A are piled up. If Y_i is the centre of an annulus that covers such a point X, then the distance XY_i must be between 2 and 3.

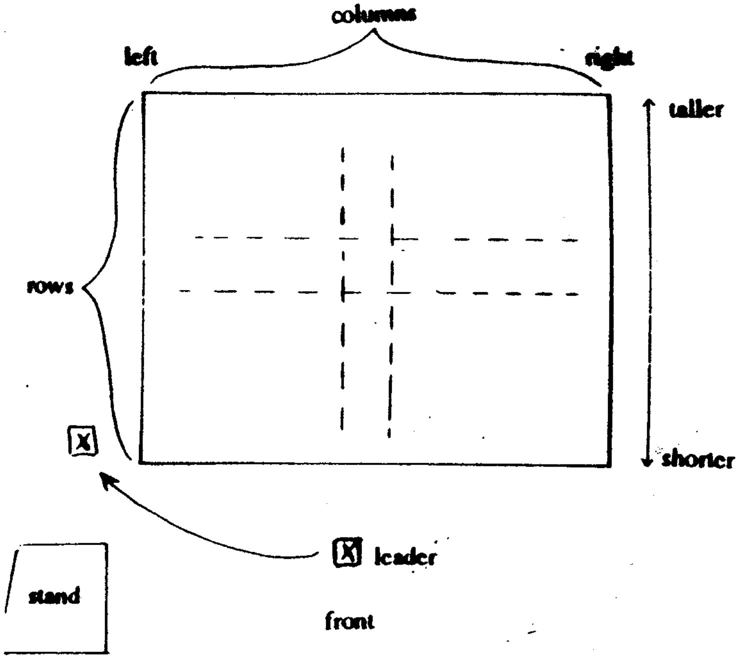


Now we can turn things around and centre a copy A^* of A at X instead of at Y_i. Then A^* covers Y_i.

Since at least 10 copies of A cover X, the special annulus A^* centered at X covers their 10 (or more) centres $Y_1, Y_2, \cdots, Y_{10}$, each of which belongs to the set S. See [8].

Example 8: **Marching Band**

When the leader of a marching band faced his musicians, he saw that some of the shorter people were hidden in the pack behind taller players. Keeping the columns intact, he brought the shorter ones forward till the people in each column stood in nondecreasing order of height from front to back.

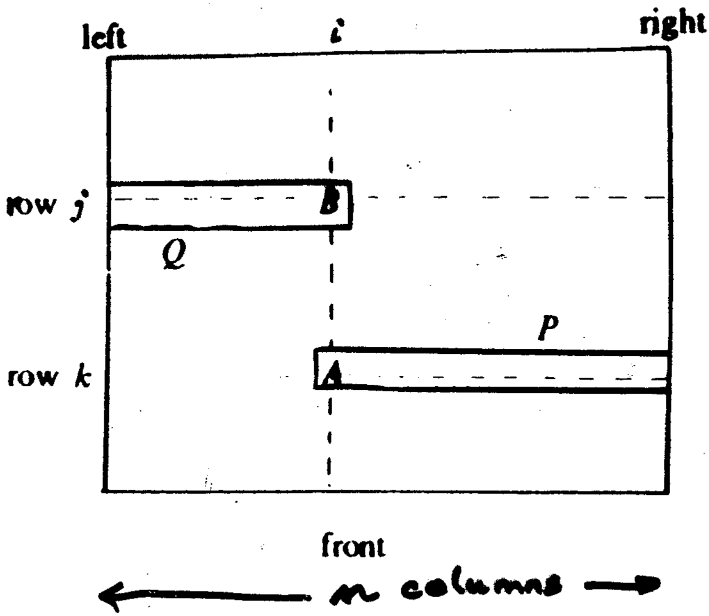


Later, they were to salute the dignitaries in a reviewing stand which they

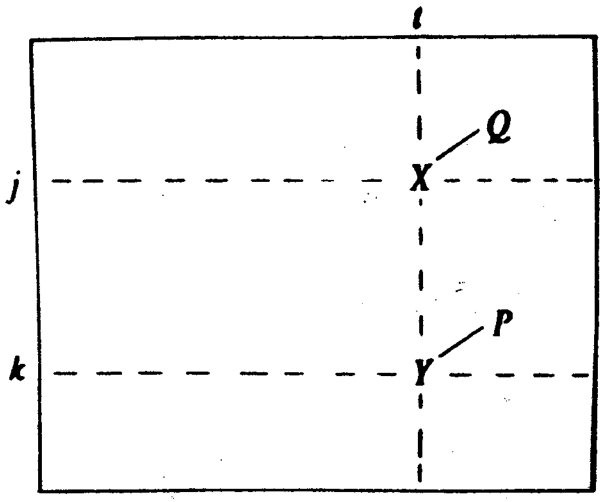
would pass on their right, so the bandmaster went around to see how they looked from the side. He found that some of the shorter players were again blocked from view.
To correct this, he did to the rows what he had just done to the columns: keeping the rows intact, he arranged the players within each row in nondecreasing order of height from left to right (that is, from his *left to right as he faced the troupe).*
In fact, he had no need to worry that this shuffling about within the rows would foul up his carefully ordered columns.

For suppose that after both rearrangements, we find, in column i, person A who is both taller and closer to the front than person B. We know that in row j, everyone in segment Q is no taller than person B, and in row k, everyone in segment P is no shorter than person A. Since A is taller than B, everyone in P is taller than everyone in Q.

It is important to note here that the total number of people in P and Q is $n + 1$.



Now we reverse the rearrangement of the rows, so that we are back to the stage where the columns were ordered but the rows still needed ordering.
In row j the elements of Q are put into their former places in the row, and in row k the elements of P are sorted into theirs. Since we have a total of $n+1$ people to replace in their original columns, the Pigeonhole Principle tells us that two of them must finish up in the same column, say, column ℓ.



At this intermediate stage, the columns are properly sorted, with X at least as tall as Y. But $X \in Q$ and $Y \in P$ so Y must be taller than X.

Now we have a contradiction: it arose from assuming that the row rearrangement must have disturbed the column ordering. So, in fact, the row rearrangement caused no disturbance of the column ordering.

This example parallels one that arises in considering the properties of Young Tableaux; see [9].

Now we look at our second basic technique, namely, **Two-way Counting**, which simply says that counting the number of elements of a set in two different ways gives the same result, a fact which is often used in combination with the Pigeonhole Principle. A convenient statement of this idea is the following:
Summing the entries in a matrix by rows or by columns gives the same result.
(However, at the time this was written, most references on the web to two-way counting concerned Enron, Arthur Anderson, and the fact that counting the same thing in several different ways gives as many different results as convenient.)

Example 9: **Sums of numbers**
If the nine non-negative real numbers

$$a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9$$

sum to 90, *then there must be four of them with sum at least* 40.

We write the nine numbers four times, arranged as shown in the table. Thus each row of the table sums to 90, and the whole table must sum to 360. But now each of the nine columns must sum to at least $360/9 = 40$.

$$\begin{array}{ccccccccc} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & a_9 \\ a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & a_9 & a_1 \\ a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & a_9 & a_1 & a_2 \\ a_4 & a_5 & a_6 & a_7 & a_8 & a_9 & a_1 & a_2 & a_3 \end{array}$$

Example 10: **Divisors of positive integers**
For every positive integer n, let $d(n)$ be the number of positive integers that divide n. For instance:

$d(15) = 4$, with divisors 1, 3, 5, 15;
$d(16) = 5$, with divisors 1, 2, 4, 8, 16;

$$d(17) = 2, \text{ with divisors } 1,\ 17;$$
$$d(18) = 6, \text{ with divisors } 1,\ 2,\ 3,\ 6,\ 9,\ 18.$$

Again $d(41) = 2, d(42) = 8, d(43) = 2$.
The function $d(n)$ keeps fluctuating, but we can use two-way counting to get an idea of its average behaviour, namely:
The average number of divisors of the first n positive integers,

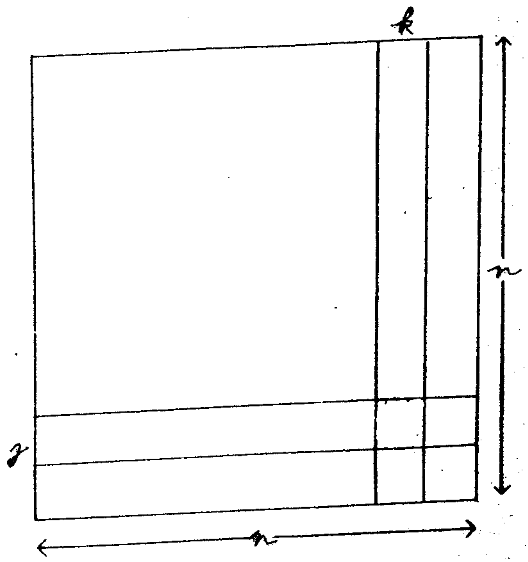
$$\frac{d(1) + d(2) + d(3) + \cdots + d(n)}{n}$$

is approximately $\log_e n$.

We think of $d(k)$ as the number of pairs (j, k) such that $j|k$. In an $n \times n$ array, suppose that

$$(j, k) = \begin{cases} 1 & \text{if } j|k, \\ 0 & \text{otherwise.} \end{cases}$$

Then the sum $d(1) + d(2) + \cdots + d(n)$ is the sum of the entries in the array, counted column by column.



Now we sum by rows instead. How many 1s are there in row j ? We want the number of multiples of j between 1 and n. This is approximately n/j (or, more precisely, between n/j and $(n/j)-1$). So the sum of the entries in the array is roughly

$$\frac{n}{1}+\frac{n}{2}+\cdots+\frac{n}{n}=n(\frac{1}{1}+\frac{1}{2}+\cdots+\frac{1}{n}).$$

But now

$$\frac{d(1)+d(2)+\cdots+d(n)}{n}$$

is roughly

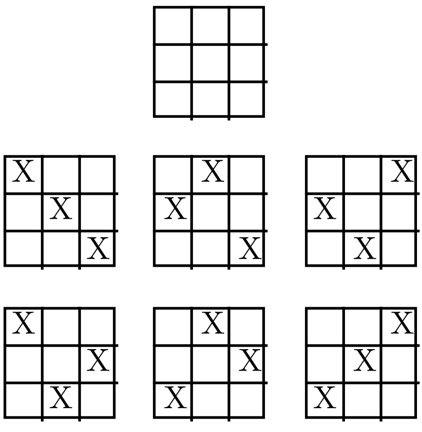
$$\frac{n\times\log_e n}{n}=\log_e n$$

since the sum by rows equals the sum by columns. See [14].

Our next example needs the idea of a **transversal**, that is, a set of n cells, one in each row and one in each column, in an $n\times n$ square. In such a square there are

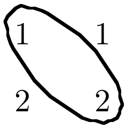
$$n\times(n-1)\times(n-2)\times\cdots\times 1=n!$$

transversals. The diagram shows the possible transversals in a 3×3 square.



Example 11: **Transversals with distinct numbers**
Let $n \geq 4$ be even. In any $n \times n$ square in which each of $n^2/2$ numbers appears twice, there is a transversal without duplication.

Certainly $n = 2$ is an exception.



1	2
2	1

1	7	6	3
1	2	4	6
5	8	3	5
7	8	4	2

as shown in the top right. The 4×4 that shows a transversal without duplication. The main diagonal has a duplication but the transversal with cells circled does not.

Note: if the two cells occupied by a given number are always in the same row or the same column, then no transversal has any duplication. Trouble only starts when a pair of cells occupied by the same number are not in the same row or the same column.
We call a pair of cells in different rows and different columns, but containing the same number, a **singular pair**.

- *Any singular pair is contained in precisely* $(n - 2)!$

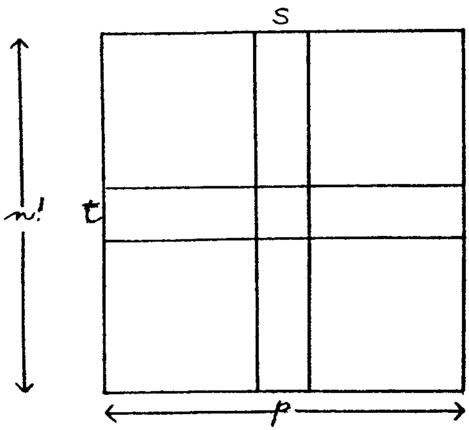
transversals. For if we choose a singular pair, then we can choose the next element in $n-2$, the next in $n-1$ ways, and so on.

- *Every transversal contains at least $n/2$ distinct numbers,* since no number occurs more than twice in the square.
- *If there are p singular pairs, then* $0 < p < n^2/2$.

Suppose a given $n \times n$ square has p singular pairs. We take an array with $n!$ rows, one for each transversal, and p columns, one for each singular pair. If t is a transversal and s a singular pair, then:

$$(t, s) = \begin{cases} 1 & \text{if } t \text{ contains } s, \\ 0 & \text{otherwise} \end{cases}$$

and we sum the entries in two ways.



Summing by columns, the total is

$$(p) \times (n-2)!$$

and summing by rows, the total is

$$(n!) \times \mathcal{S}$$

where $\mathcal{S}$ is the average number of singular pairs per transversal. Then

$$\mathcal{S} = \frac{p(n-2)!}{n!}$$

and, since $p \leq n^2/2$,

$$\mathcal{S} \leq \frac{n^2}{2} \times \frac{(n-2)!}{n!} = \frac{n}{2(n-1)}.$$

But $n \geq 4$ which means that

$$\mathcal{S} < 1$$

and there must be a transversal with less than one singular pair, that is, with no duplication.

This kind of argument, combining the Pigeonhole Principle with Two-way Counting, is sometimes referred to as 'existence by averaging' [14], [15], [10], and is the source of many interesting problems. We look briefly at examples involving a *latin square of order* n, that is, an $n \times n$ square based on the set $\{1, 2, \ldots, n\}$, in which each element appears precisely once in each row and in each column. We use the fact that

$$(1 - \frac{1}{q})^q \rightarrow \frac{1}{e} \approx 0.37.$$

An argument similar to that of Example 11 shows that the average number of distinct symbols in the transversals of a latin square of order n must be

$$n(1 - \frac{1}{2!} + \frac{1}{3!} - \ldots \pm \frac{1}{n!}),$$

so there is a transversal with at least $(1 - \frac{1}{e})n \approx 0.63n$ elements.

Now suppose that $qm = n$. Then in a latin square of order n, there exist q rows such that the union of the sets of integers in the first m columns of these q rows contains at least

$$n[1 - (1 - \frac{1}{q})^q]$$

distinct elements. For instance, if $m = 2$ and $q = 3$, then in a latin square of order 6, there are 3 rows such that the first 2 columns of these rows contain at least

$$6[1-(1-\frac{1}{3})^3] = 38/9$$

distinct elements. In other words, they contain at least 5 distinct elements; see [15].

1	2	3	4	5	6
2	3	1	5	6	4
3	1	2	6	4	5
4	6	5	2	1	3
5	4	6	3	2	1
6	5	4	1	3	2

In the 6×6 latin square, rows 1, 2 and 4 contain all the elements except 5 in columns 1 and 2.

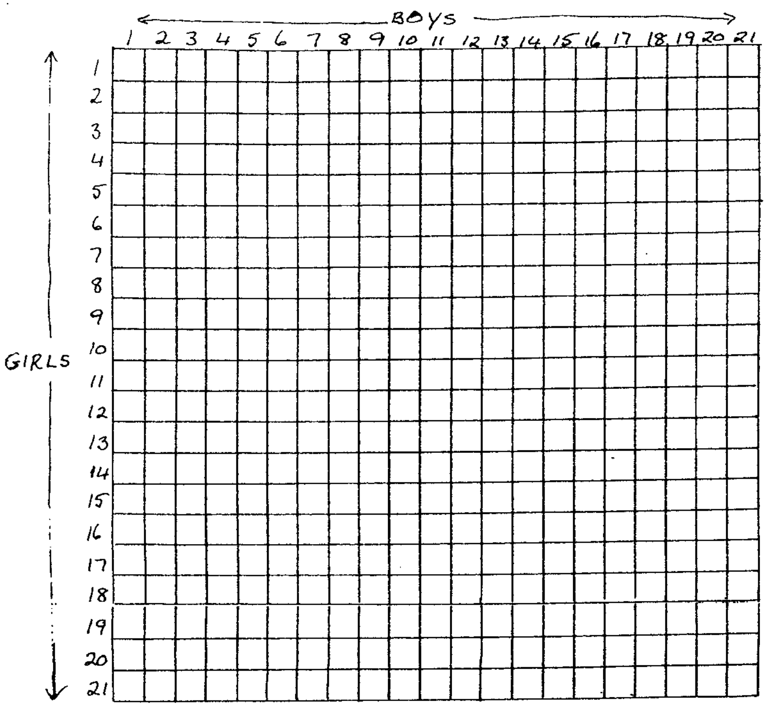
Even more interesting is the *Ryser conjecture:*
In a latin square of order n, there is a transversal with n elements if n is odd, and a transversal with $n-1$ elements, if n is even. See also [2],[12], [13], [16].

It seems appropriate to conclude a collection of problems from mathematics competitions with a competition problem concerning a mathematics competition, namely, one from the 2001 IMO. The solution given here was due to Reid Barton, a gold medallist ([4]).

Example 12: **Mathematics Competition**
21 *girls and* 21 *boys took part in a mathematical contest. Each contestant solved at most six problems. For each girl and each boy (that is, for each boy–girl pairing) at least one problem was solved by both of them.*

Then there must be a problem that was solved by at least three girls and at least three boys.

To see this, take a 21×21 array, with one row for each girl and one column for each boy. In each cell of the array, place a letter representing a problem that was solved by the girl in the corresponding row and the boy in the corresponding column. Then every cell of the array will be filled.



No row can contain more than six different letters, so each row contains repeated letters. Look for letters that appear at least three times in a row, and colour every cell of the row containing those letters red.
How many cells in each row are colored red?
Each cell *not* colored red is filled with a letter that occurs at most twice

in the row. At most five letters can occur at most twice, so at most 10 cells are not colored red. Thus the number of red cells in each row is at least 11, so
more than half the cells in each row are red.

A similar argument works for the boys, where we colour cells in each column blue to represent problems solved by at least three girls. So
more than half the cells in each column are blue.

Since more than half the cells are red and more than half are blue, there must be at least one cell colored *both* red and blue.
The letter in this cell represents a problem that was solved by at least three girls and at least three boys.

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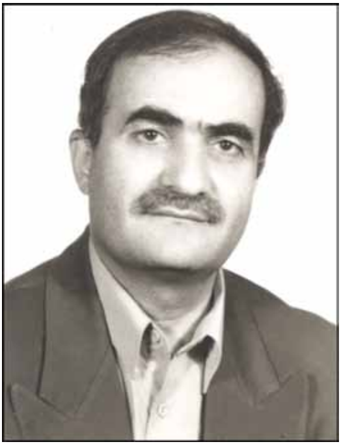
Mathematics Competitions, Mathematics Teachers and Mathematics Education in Iran

Ali Rejali



Ali Rejali is Associate Professor of Mathematics and Statistics at Isfahan University of Technology, Iran. He is the Cofounder of the Iranian Mathematics Competitions, founder of Isfahan Mathematics House and Societies for Teachers of Mathematics in Iran and the Iranian Association for Mathematics Teachers' Societies. He is also a member of the Awards Committee of the World Federation of National Mathematics Competitions.

We had many goals for improvement of mathematics education when we started the mathematics competition in Iran, but due to some changes on the rules, most of them have not been achieved. One of them was the improvement of mathematics knowledge, and problem solving skills among teachers. The results of a study on the involvement of mathematics teachers in competitions, as well as the positive and negative effects of the competitions on education in Iran, is presented.

Introduction

High school mathematics competitions in Iran started in 1984 [1]. One of the goals was to involve the teachers, schools, cities and provinces as well as students in competition type activities for improvement of mathematics education throughout the country [2].

In order to achieve this goal, we started competitions in schools, cities and provinces. The teachers were involved in all activities related to competitions [3]. But later on due to a need for the success of the Iranian teams in IMOs, they have changed the rules.

Now a two stage national competition is held each year. On the first stage, through a multiple choice national test, a central committee chooses the winners of the first round. Then there is another competition among the winners of the first round, through which the committee chooses the winners of the second round. These students take part in a series of training workshops, and the IMO team will be chosen from the best performers of these students.

Under this policy, the teachers are less involved, and many students do not take part in competitions as they feel unsuccessful if they have not achieved high results in these competitions, and, as a result, there are few benefits to mathematics education in general from these activities.

In order to test above hypothesis, we ran an opinion poll study among all mathematics teachers throughout the country, with the help of the Iranian Association of Mathematics Teachers' Societies. What follows is the procedure and the results:

Sampling Procedure

Cluster sampling is being used with 121 teachers from the list of the members of different Mathematics Teachers' Societies throughout the country, (these teachers are more aware of the news and problems about mathematics education).

Here, the clusters are the provinces, each listing unit consists of members of Mathematics Teachers' Society of that province. Elementary unit is a typical high school mathematics teacher, as a member of the corresponding society [4]. Sample size ($n = 121$) is obtained by estimating the overall sample size with the use of the variance for similar study in Iran and 10 percent accepted error.

The number of elementary units in each province was calculated relative to the number of population of that province, (using 1996 Iranian Census). A tested questionaire was designed for this opinion poll, and follow up attempts have been made.

Results

- Only 65.3% of these mathematics teachers are familiar with the procedure of mathematics competitions.
- Only 16.5% of them agree with the procedure. (Some small provinces with low level of mathematics education agree more.)
- Most of them believe that mathematics competitions should be more popular.
- 54.5% of the teachers believe that in general mathematics competitions have positive effects on mathematics education.
- 57.9% believe that the competitions have positive effect on solving the problem of lack of interest in mathematics.
- 56.2% believe that the competitions stimulate students in learning more mathematics.
- 50.4% believe that they encourage students in solving problems.
- 74% believe that mathematics education will benefit from mathematics competitions.
- But 76% of these teachers believe that this competition does not belong to all students, and 58.7% believe that it should be designed so that most students can be involved in it.

For the involvement of teachers in running competitions,

- only 39.7% of teachers believe that they are involved in making questions, 59.5% are involved in choosing students to participate in Iranian mathematics competitions from schools, only 33.9% believe that the teachers are involved in choosing winners in cities, 9.9% in provinces and 2.5% in the country (National Iranian Mathematics Olympiad). Even for organizing the competitions as observers only 19% are involved.

- Although only 31.4% believe that mathematics competitions have positive effect on mathematics education of all students, most believe that it stimulates students in general.
- On the other hand, only 9.9% believe that it has negative side effects on students, due to discouragement from not being winner.
- 33.4% believe that it is a high school business, and only 24.8% believe that it is a university business.
- 28.9% believe that the competitions have positive effects on social behaviour of the students. 44.6% have no idea on this question.
- Only 11.6% believe that the competitions have negative effects on social behaviour of students, due mostly to discouragement.
- 52.1% believe that separating the so called gifted students is not a suitable activity for mathematics education.
- 54.5% believe that mixing up students has positive effects on the improvement of education.
- 57.9% disbelieve the method of choosing gifted students. Some of them (low%) believe that the teachers are capable of choosing these students throughout the classes.
- 53.7% believe in the positive effects of competitions on winners, and 35.5% in the negative effects of the competitions on non winners.
- Most of the teachers believe that competitions should be more popular, and we should prepare all students, even the students from low income families to take part in them.
- Most of the teachers agree with the author for the need for more involvement of teachers in competitions.

The overall results are:

1. A need for more involvement of teachers.

2. A need for changing rules to have higher involvement of both teachers and students.

3. The need to prepare all students for taking part in competitions.

4. In any situation, competitions have positive effects on mathematics education [3].

The author is thankful to Miss Fathi and Miss Masaeli for working on the results, to the Iranian Association of Mathematics Teachers' Societies, and all Societies for Mathematics Teachers in Iran, for helping him in this opinion poll. Also he thanks the Iran Education Evaluation Organization for its financial support.

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* * *

Tournament of Towns Corner

Andrei Storozhev

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Selected Problems from Tournament 24

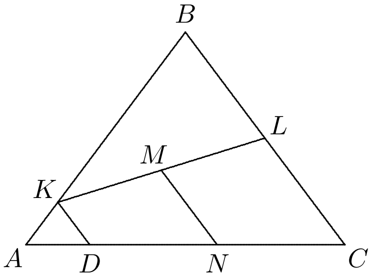
In Tournament 24 the Problems Committee challenged students to solve a wide range of problems: from polynomial equations to three-dimensional geometric inequalities. In the second round, both Junior and Senior O Level papers consisted of five problems, and Junior and Senior A Level papers were made up of six and seven problems respectively. Here are selected questions with solutions from this round of the Tournament.

1. We put 2003 dollars in several purses, and the purses in several pockets. The number of dollars in any pocket is less than the total number of purses. Is it necessarily true that the number of dollars in some purse is less than the total number of pockets?

Solution. Let m be the total number of pockets and n be the total number of purses. Let x be the maximum number of dollars in any pocket and y be the minimum number of dollars in any purse. Then $ny \leq 2003 \leq mx$. It follows that if $n > x$, then we must have $m > y$.

2. In triangle ABC, $AB = BC$. K is a point on AB and L a point on BC such that $AK + LC = KL$. A line through the midpoint M of KL and parallel to BC intersects AC at the point N. Determine $\angle KNL$.

Solution. Through K, draw a line parallel to BC, cutting AC at D.



Since triangles AKD and ABC are similar, we have $AK = KD$. Since $KM = ML$. we have

$$MN = \frac{KD + LC}{2} = \frac{AK + LC}{2} = \frac{KL}{2} = KM = ML.$$

Hence N lies on the semicircle with diameter KL, so that $\angle KNL = 90°$.

3. Is it possible to cover a 2003×2003 board using horizontal 1×2 rectangles and vertical 3×1 rectangles?

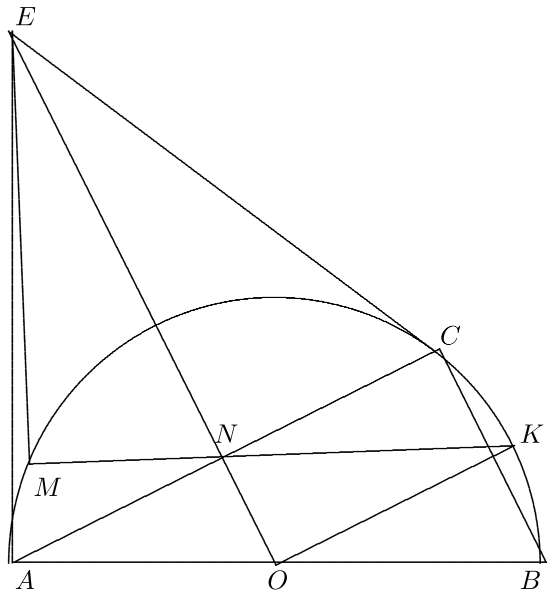
Solution. Colour the vertical columns of the board alternately black and white. Initially, the difference between the numbers of black and white squares that are uncovered is 2003. Each horizontal 1×2 rectangle covers one square of each colour, and cannot affect this difference. Each vertical 3×1 rectangle covers three squares of the same colour, and can either raise or lower this difference by 3. Since 2003 is not a multiple of 3, not all squares can be covered.

4. Joannie has a block of chocolate in the shape of an equilateral triangle of side n, divided into n^2 equilateral triangles of side 1 by lines parallel to the sides of the block. She shares it with Petra by playing the following game. Joannie begins by breaking off a triangular piece of any size along one of the lines, eats it and passes what remains of the block to Petra. Each takes turns passing the block of chocolate back and forth. If at any time, it is not possible to break off a triangular piece, the player whose turn it is loses immediately. The piece eaten must be an equilateral triangle of side 1, and whoever eats that wins the game. For each n, determine whether Joannie or Petra has a winning strategy.

Solution. Joannie cannot eat the whole block unless $n = 1$, in which case she wins immediately. For $n \geq 2$, she must leave behind an isosceles trapezoid. Petra must reduce it to a parallelogram. If she leaves behind a pentagon, Joannie can leave her a hexagon from which no moves are possible. For the same reason, Joannie must change the parallelogram back to an isosceles trapezoid. Eventually, the parallelogram Petra leaves is actually a rhombus, which Joannie then converts to an isosceles triangle T. If T has side 1, then Petra wins. If n is a prime, the first parallelogram Petra products will have two sides that are relatively prime to each other, so that T will have side 1. Thus Petra wins whenever n is a prime. Suppose n is composite. Then Joannie can make her first isosceles trapezoid to have equal legs of length equal to the smallest prime divisor p of n. Then T will have side p. Since it is Petra's turn to move, Joannie wins. In summary, Petra wins if and only if n is prime.

5. C is a point on the circle with diameter AB. K is the midpoint of the arc BC not containing A, N is the midpoint of the segment AC, and the line KN intersects the circle again at M. Tangents to the circle at the points A and C intersect at the point E. Prove that $\angle EMK = 90°$.

Solution. Let O be the centre of the circle Then OK is perpendicular to BC and OE is perpendicular to AC. Since $AN = NC$ and $AO = OB$, ON is parallel to BC, and hence perpendicular to AC. It follows that E, N and O are collinear. Note that both $AECO$ and $AMCK$ are cyclic quadrilaterals. Hence $MN \cdot NK = AN \cdot NC = EN \cdot NO$, so that $EMOK$ is also a cyclic quadrilateral. It follows that $\angle EMK = \angle EOK = 90°$.



6. John chooses an integer greater than 100. Mary calls out an integer d greater than 1. If John's integer is divisible by d, then Mary wins. If not, John subtracts d from his number and the game continues. Mary is not allowed to call out any number twice. When John's number becomes negative, Mary loses. Can Mary play in such a way as to always win?

Solution. Mary can guarantee a win by calling in succession 2, 3, 4, 6, 16 and 12. Let John's current number be congruent modulo 12 to k, $0 \leq k \leq 11$.

If $k = 0, 2, 4, 6, 8, 10$, Mary wins after calling 2.
If $k = 5, 11$, Mary wins after calling 3.
If $k = 1, 9$, Mary wins after calling 4.
If $k = 3$, Mary wins after calling 6.

If $k = 7$, after calling 2, 3, 4 and 6 Mary obtains a number congruent to 4 modulo 12. Therefore either Mary wins on her next move after calling 16 or after that by calling 12.

The maximum amount that could have been subtracted from John's original number is 2+3+4+6+16+12=43. Since it is over 100, Mary has won before the current value of John's number drops below 0.

7. A triangle has circumradius R and inradius r. If a is the length of the longest side while h is the length of the shortest altitude, prove that $\frac{R}{r} > \frac{a}{h}$.

Solution. We have $2R > a$ since no side can be longer than the diameter of the circumcircle. On the other hand, no altitude can be shorter than the diameter of the incircle, so that $2r < h$. Division yields the desired result $\frac{R}{r} > \frac{a}{h}$.

World Wide Web

Information on the Tournament, how to enter it, and its rules are on the World Wide Web. Information on the Tournament can be obtained from the Australian Mathematics Trust web site at

http://www.amt.canberra.edu.au

Books on Tournament Problems

There are four books on problems of the Tournament available. Information on how to order these books may be found in the Trust's advertisement elsewhere in this journal, or directly via the Trust's web page.

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Over 250 pages of challenging questions and solutions from the Australian Mathematics Competition papers from 1985-1991.

Australian Mathematics Competitions Book 3 1992-1998
W Atkins, JE Munro and PJ Taylor

More challenging questions and solutions from the Australian Mathematics Competition papers from 1992-1998.

Australian Mathematics Competition Book 3 on CD
Programmed by E. Storozhev

This CD contains the same problems and solutions as in the corresponding book. The problems can be accessed in topics as in the book and in this mode is ideal to help students practice particular skills. In another mode students can simulate writing one of the actual papers and determine the score that they would have gained. The CD runs on all Windows platforms.

Problem Solving Via the AMC
Edited by Warren Atkins

This 210 page book consists of a development of techniques for solving approximately 150 problems that have been set in the Australian Mathematics Competition. These problems have been selected from topics such as Geometry, Motion, Diophantine Equations and Counting Techniques.

Methods of Problem Solving, Book 1
Edited by JB Tabov, PJ Taylor

This introduces the student aspiring to Olympiad competition to particular mathematical problem solving techniques. The book contains formal treatments of methods which may be familiar or introduce the student to new, sometimes powerful techniques.

Methods of Problem Solving, Book 2
JB Tabov & PJ Taylor

After the success of Book 1, the authors have written Book 2 with the same format but five new topics. These are the Pigeon-Hole Principle, Discrete Optimisation, Homothety, the AM-GM Inequality and the Extremal Element Principle.

Mathematical Toolchest
Edited by AW Plank & N Williams

This 120 page book is intended for talented or interested secondary school students, who are keen to develop their mathematical knowledge and to acquire new skills. Most of the topics are enrichment material outside the normal school syllabus, and are accessible to enthusiastic year 10 students.

International Mathematics — Tournament of Towns (1980-1984)
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Edited by PJ Taylor

International Mathematics — Tournament of Towns (1993-1997)
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The International Mathematics Tournament of the Towns is a problem solving competition in which teams from different cities are handicapped according to the population of the city. Ranking only behind the International Mathematical Olympiad, this competition had its origins in Eastern Europe (as did the Olympiad) but is now open to cities throughout the world. Each book contains problems and solutions from past papers.

Challenge! 1991 – 1995
Edited by JB Henry, J Dowsey, A Edwards, L Mottershead, A Nakos, G Vardaro

The Mathematics Challenge for Young Australians attracts thousands of entries from Australian High Schools annually and involves solving six in depth problems over a 3 week period. In 1991-95, there were two versions – a Junior version for Year 7 and 8 students and an Intermediate version for Year 9 and 10 students. This book reproduces the problems from both versions which have been set over the first 5 years of the event, together with solutions and extension questions. It is a valuable resource book for the class room and the talented student.

USSR Mathematical Olympiads 1989 – 1992
Edited by AM Slinko

Arkadii Slinko, now at the University of Auckland, was one of the leading figures of the USSR Mathematical Olympiad Committee during the last years before democratisation. This book brings together the problems and solutions of the last four years of the All-Union Mathematics Olympiads. Not only are the problems and solutions highly expository but the book is worth reading alone for the fascinating history of mathematics competitions to be found in the introduction.

Australian Mathematical Olympiads 1979 – 1995
H Lausch & PJ Taylor

This book is a complete collection of all Australian Mathematical Olympiad papers since the first competition in 1979. Solutions to all problems are included and in a number of cases alternative solutions are offered.

Chinese Mathematics Competitions and Olympiads 1981-1993
A Liu

This book contains the papers and solutions of two contests, the Chinese National High School Competition from 1981-82 to 1992-93, and the Chinese Mathematical Olympiad from 1985-86 to 1992-93. China has an outstanding record in the IMO and this book contains the problems that were used in identifying the team candidates and selectiing the Chinese teams. The problems are meticulously constructed, many with distinctive flavour. They come in all levels of difficulty, from the relatively basic to the most challenging.

Asian Pacific Mathematics Olympiads 1989-2000
H Lausch & C Bosch-Giral
With innovative regulations and procedures, the APMO has become a model for regional competitions around the world where costs and logistics are serious considerations. This 159 page book reports the first twelve years of this competition, including sections on its early history, problems, solutions and statistics.

Polish and Austrian Mathematical Olympiads 1981-1995

ME Kuczma & E Windischbacher

Poland and Austria hold some of the strongest traditions of Mathematical Olympiads in Europe even holding a joint Olympiad of high quality. This book contains some of the best problems from the national Olympiads. All problems have two or more independent solutions, indicating their richness as mathematical problems.

Seeking Solutions

JC Burns

Professor John Burns, formerly Professor of Mathematics at the Royal Military College, Duntroon and Foundation Member of the Australian Mathematical Olympiad Committee, solves the problems of the 1988, 1989 and 1990 International Mathematical Olympiads. Unlike other books in which only complete solutions are given, John Burns describes the complete thought processes he went through when solving the problems from scratch. Written in an inimitable and sensitive style, this book is a must for a student planning on developing the ability to solve advanced mathematics problems.

101 Problems in Algebra
from the Training of the USA IMO Team
Edited by T Andreescu & Z Feng

This book contains one hundred and one highly rated problems used in training and testing the USA International Mathematical Olympiad team. These problems are carefully graded, ranging from quite accessible towards quite challenging. The problems have been well developed and are highly recommended to any student aspiring to participate at National or International Mathematical Olympiads.

Mathematical Contests – Australian Scene
Edited by AM Storozhev, JB Henry & DC Hunt

These books provide an annual record of the Australian Mathematical Olympiad Committee's identification, testing and selection procedures for the Australian team at each International Mathematical Olympiad. The books consist of the questions, solutions, results and statistics for: Australian Intermediate Mathematics Olympiad (formerly AMOC Intermediate Olympiad), AMOC Senior Mathematics Contest, Australian Mathematics Olympiad, Asian-Pacific Mathematics Olympiad, International Mathematical Olympiad, and Maths Challenge Stage of the Mathematical Challenge for Young Australians.

WFNMC — Mathematics Competitions
Edited by Warren Atkins

This is the journal of the World Federation of National Mathematics Competitions (WFNMC). With two issues each of approximately 80-100 pages per year, it consists of articles on all kinds of mathematics competitions from around the world.

Parabola

This Journal is published in association with the School of Mathematics, University of New South Wales. It includes articles on applied mathematics, mathematical modelling, statistics, and pure mathematics that can contribute to the teaching and learning of mathematics at the senior secondary school level. The Journal's readership consists of mathematics students, teachers and researchers with interests in promoting excellence in senior secondary school mathematics education.

ENRICHMENT STUDENT NOTES

The Enrichment Stage of the Mathematics Challenge for Young Australians (sponsored by the Dept of Education, Science and Training) contains formal course work as part of a structured, in-school program. The Student Notes are supplied to students enrolled in the program along with other materials provided to their teacher. We are making these Notes available as a text book to interested parties for whom the program is not available.

Newton Enrichment Student Notes
JB Henry

Recommended for mathematics students of about Year 5 and 6 as extension material. Topics include polyominoes, arithmetricks, polyhedra, patterns and divisibility.

Dirichlet Enrichment Student Notes
JB Henry

This series has chapters on some problem solving techniques, tessellations, base five arithmetic, pattern seeking, rates and number theory. It is designed for students in Years 6 or 7.

Euler Enrichment Student Notes
MW Evans and JB Henry

Recommended for mathematics students of about Year 7 as extension material. Topics include elementary number theory and geometry, counting, pigeonhole principle.

Gauss Enrichment Student Notes
MW Evans, JB Henry and AM Storozhev

Recommended for mathematics students of about Year 8 as extension material. Topics include Pythagoras theorem, Diaphantine equations, counting, congruences.

Noether Enrichment Student Notes
AM Storozhev

Recommended for mathematics students of about Year 9 as extension material. Topics include number theory, sequences, inequalities, circle geometry.

Pólya Enrichment Student Notes
G Ball, K Hamann and AM Storozhev

Recommended for mathematics students of about Year 10 as extension material. Topics include polynomials, algebra, inequalities and geometry.

T-SHIRTS

T-shirts celebrating the following mathematicians are made of 100% cotton and are designed and printed in Australia. They come in white, and sizes Medium (Polya and Newton only) and XL.

Leonhard Euler T–shirt

The Leonhard Euler t-shirt depicts a brightly coloured cartoon representation of Euler's famous Seven Bridges of Königsberg question.

Carl Friedrich Gauss T–shirt

The Carl Friedrich Gauss t-shirt celebrates Gauss' discovery of the construction of a 17-gon by straight edge and compass, depicted by a brightly coloured cartoon.

Emmy Noether T–shirt

The Emmy Noether t-shirt shows a schematic representation of her work on algebraic structures in the form of a brightly coloured cartoon.

George Pólya T–shirt

George Pólya was one of the most significant mathematicians of the 20th century, both as a researcher, where he made many significant discoveries, and as a teacher and inspiration to others. This t-shirt features one of Pólya's most famous theorems, the Necklace Theorem, which he discovered while working on mathematical aspects of chemical structure.

Peter Gustav Lejeune Dirichlet T–shirt

Dirichlet formulated the Pigeonhole Principle, often known as Dirichlet's Principle, which states: "If there are p pigeons placed in h holes and p>h then there must be at least one pigeonhole containing at least 2 pigeons." The t-shirt has a bright cartoon representation of this principle.

Alan Mathison Turing T-shirt

The Alan Mathison Turing t-shirt depicts a colourful design representing Turing's computing machines which were the first computers.

Sir Isaac Newton T-shirt

The T Shirt features Sir Isaac Newton together with an apple which is claimed, after falling from a tree, inspired Sir Isaac to discover the laws of motion and gravity which bear his name. Sir Isaac made a major contribution to mathematics as co-developer of the Calculus. This is honoured in the T shirt by showing acceleration (*s* with two dots on top) as the second derivative of distance. The law depicted in the equation is known as Newton's second law of motion, Force equals mass times acceleration.

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