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The aims of the Federation are:–

1. ***to promote excellence in, and research associated with, mathematics education through the use of school mathematics competitions;***
2. ***to promote meetings and conferences where persons interested in mathematics contests can exchange and develop ideas for use in their countries;***
3. ***to provide opportunities for the exchanging of information for mathematics education through published material, notably through the Journal of the Federation;***
4. ***to recognize through the WFNMC Awards system persons who have made notable contributions to mathematics education through mathematical challenge around the world;***
5. ***to organize assistance provided by countries with developed systems for competitions for countries attempting to develop competitions;***
6. ***to promote mathematics and to encourage young mathematicians.***

From the President

We are quite close to the major event for the World Federation of National Mathematics Competitions (WFNMC) in 2006. This is its fifth Conference which is to take place in Cambridge, England, from July 22$^{\text{nd}}$ to July 28$^{\text{th}}$. The Federation today is a well established organization. Its activities are regulated by a constitution and by a policy statement on competitions and mathematics education. The Federation publishes this journal and gives Paul Erdös Awards. For more than 20 years it has provided support for those interested in and involved with mathematics competitions. The future of the organization however depends on its ability to identify and to respond to the numerous challenges accompanying its major activities. With this in mind I would like to draw your attention to two particular areas.

In many countries, year after year, some schools consistently "produce" more competition winners than other schools. What is the reason behind this phenomenon? Why are some schools more successful than others?

The reasons may be numerous and fairly different in nature. Very often, however, the prominent success of a particular school can be attributed to the dedicated efforts of a single teacher or a small group of teachers. For these excellent teachers, teaching is a vocation, a mission, and not just a means to make ends meet. Such special teachers are real assets for the school and for the whole country. They possess both the necessary scientific ability and the extraordinary personality needed to identify and motivate for hard work the future winners in competitions.

Such teachers need special care. Their higher scientific ability is acquired very slowly, at the expense of great personal efforts. It is no secret that the success of these teachers depends very strongly on their working environment and on the appreciation by their colleagues and by the school administration. Very often however the actual working conditions in schools do not support the work and the development of these dedicated teachers.

There is a lot that can (and will have to) be done in order to improve this situation. For instance, the materials available to teachers should not include problems and solutions only. They should also provide didactical instructions to guide teachers in how to use these materials in their work with higher ability students, and indicate what type of reactions and difficulties to expect on the part of their students. For this to happen special research and experimentation is needed, conducted with the participation of professional math educators. The World Federation of National Mathematics Competitions and its Journal *Mathematics Competitions* can help a lot in this direction. Many organizations which are involved with competitions are also organizing seminars and workshops for teachers. There is valuable experience in many countries in the work of such special teachers. The positive results and problems encountered could be discussed and evaluated with the aim of disseminating good practice. Teachers are the major human resource for the development of competitions and related activities.

Another problem is that often competition-like activities in many universities and scientific institutions are not "at home" (and therefore not appreciated) both in mathematics departments (because "they concern elementary mathematics") and in mathematics education departments (because they are "too mathematical and refer to the relatively small group of talented students"). On the other hand, competition-like activities attract talented young people to possible professional careers in mathematics and in this way positively influence the development of science. Competitions also enhance the educational process. Can WFNMC do something so that both communities (research mathematicians and mathematics educators) understand their joint interest in supporting competitions and competitions-related activities?

Petar S. Kenderov
President of WFNMC
June 2006

From the Editor

Welcome to *Mathematics Competitions* Vol. 19, No 1.

At first I would like to thank the Australian Mathematics Trust for continued support, without which each issue of the journal could not be published, and in particular Heather Sommariva, Bernadette Webster and Pavel Calábek for their assistance in the preparation of this issue.

Submission of articles:
The journal *Mathematics Competitions* is interested in receiving articles dealing with mathematics competitions, not only at national and international level, but also at regional and primary school level. There are many readers in different countries interested in these different levels of competitions.

- The journal traditionally contains many different kinds of articles, including reports, analyses of competition problems and the presentation of interesting mathematics arising from competition problems. Potential authors are encouraged to submit articles of all kinds.
- To maintain and improve the quality of the journal and its usefulness to those involved in mathematics competitions, all articles are subject to review and comment by one or more competent referees. The precise criteria used will depend on the type of article, but can be summarised by saying that an article accepted must be correct and appropriate, the content accurate and interesting, and, where the focus is mathematical, the mathematics fresh and well presented. This editorial and refereeing process is designed to help improve those articles which deserve to be published.

At the outset, the most important thing is that if you have anything to contribute on any aspect of mathematics competitions at any level, local, regional or national, we would welcome your contribution.

Articles should be submitted in English, with a black and white photograph and a short profile of the author. Alternatively, the article can be submitted on an IBM PC compatible disk or a Macintosh disk. We prefere LaTeX or TeX format of contributions, but any text file will be helpful.

Articles, and correspondence, can also be forwarded to the editor by mail to

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Jaroslav Švrček,
June 2006

On a Problem of Polish Mathematical Olympiad

Józef Kalinowski

Dr. Józef Kalinowski is a senior lecturer in the Department of Mathematics at the Silesian University, Poland. He is a vice-director of the regional committee of the Mathematical Olimpiad in Katowice. He is the author of two books of problems of the Czech and Slovak Mathematics Olympiad (in Polish). He is also one of the founders and organizers of all fourteen international mathematical competitions DUEL.

There are some plane geometry problems which have many different solutions. One of them is a problem published in [1]. The solutions to this problem were established by many authors in [1]–[3].

One such problem is chosen from the first round of the LVI Polish Mathematical Olympiad in 2004/05.

Problem. *Given is an acute-angled triangle ABC in the plane. Let D be the foot of the altitude from the vertex C on the side AB. Further, let E be the foot of the perpendicular segment from the point D on the straight line BC, and finally let F be a point of the segment DE such that*

$$\frac{EF}{DF} = \frac{AD}{BD} \tag{1}$$

holds. Prove that the straight lines CF and AE are perpendicular.

This problem can be solved using methods of analytic geometry or using the algebra of vectors.

The most interesting of them are synthetic solutions of a given problem. We will present some of these solutions, based on the materials of LVIth Polish Mathematical Olympiad.

Solution 1

The triangles DBE and CDE (see Fig. 1) are similar because three corresponding angles are equal. We have $\angle DBE = \angle CDE$. Therefore,

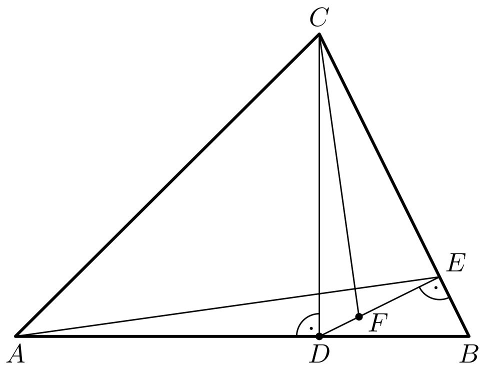

Fig. 1

$$\frac{EB}{BD} = \frac{ED}{CD}. \tag{2}$$

Multiplying both sides of (2) by $\frac{BD}{AB}$ we obtain on the left side

$$\frac{EB}{BD} \cdot \frac{BD}{AB} = \frac{EB}{AB}. \tag{3}$$

Using (1) we denote

$$\frac{EF}{DF} = \frac{AD}{BD} = k,$$

then $EF = k \cdot DF$ and $BD = \frac{1}{k} \cdot AD$. Using $ED = EF + DF$ on the right side of (2) we obtain

$$\begin{aligned}\frac{ED}{CD} \cdot \frac{BD}{AB} &= \frac{DF \cdot (k+1) \cdot \frac{1}{k} \cdot AD}{CD \cdot AB} \\ &= \frac{DF \cdot AD \cdot \left(\dfrac{\dfrac{AD}{BD} + 1}{\dfrac{AD}{BD}} \right)}{CD \cdot AB} = \frac{DF}{CD}.\end{aligned} \tag{4}$$

From (3) and (4) we have

$$\frac{BE}{AB} = \frac{DF}{CD}.$$

The triangles CDF and ABE are similar (two sides are proportional and $\angle CDF = \angle ABE$). The pairs CD, AB and EB, DF are perpendicular. Then the sides CF and AE are also perpendicular.

Solution 2

From (1) we obtain the following equivalent statements

$$\begin{aligned}\frac{EF}{DF} + 1 &= \frac{AD}{BD} + 1, \\ \frac{EF + DF}{DF} &= \frac{AD + BD}{BD}, \\ \frac{ED}{DF} &= \frac{AB}{BD},\end{aligned}$$

and then

$$ED \cdot BD = AB \cdot DF. \tag{5}$$

The triangles EBD and EDC are similar (three corresponding angles are equal). Since $\angle DBE = \angle EDC$ then

$$\frac{EB}{BD} = \frac{ED}{DC},$$

and we have

$$ED \cdot BD = EB \cdot DC. \tag{6}$$

From (5) and (6) we have

$$EB \cdot CD = AB \cdot DF,$$

$$\frac{EB}{AB} = \frac{DF}{CD}.$$

Since $\angle ABE = \angle FDC$ and two pairs of sides are proportional, then triangles ABE and CDF are similar. Using the notation from Fig. 2 we have $\angle BAE = \angle DCF$, $\angle DZA = \angle CZG$ and right-angled triangles AZD

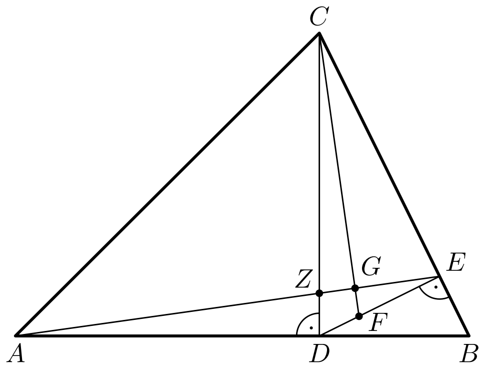

Fig. 2

and CZG are similar. Then $\angle ADZ = \angle CGZ = 90°$, i.e. $CF \perp AE$.

Solution 3

Let H be the point of the segment BC such that DE is parallel to AH (see Fig. 3). Then the proportion

$$\frac{AD}{BD} = \frac{HE}{EB}$$

holds. Using (1) we obtain the proportion

$$\frac{HE}{EB} = \frac{EF}{FD}. \tag{7}$$

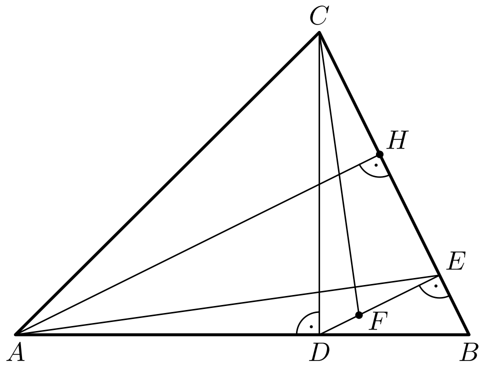

Fig. 3

The right-angled triangles CDE and ABH are similar. Then $\angle DCE = \angle BAH$. Since (7) holds, then also $\angle BAE = \angle DCF$. The straight lines CD and AB are perpendicular. We obtain the straight line CF from the straight line CD after the rotation with the centre C through the oriented angle DCF. We also obtain the straight line AE from the straight line AB after the rotation with the centre A and through the oriented angle BAE. Using these two rotations we obtain $CF \perp AE$.

Solution 4

Let H be the point of the segment BC such that DE is parallel to AH (see Fig. 4). Then the proportion

$$\frac{AD}{BD} = \frac{HE}{EB}$$

holds. Using (1) we obtain the proportion

$$\frac{HE}{EB} = \frac{EF}{DF}.$$

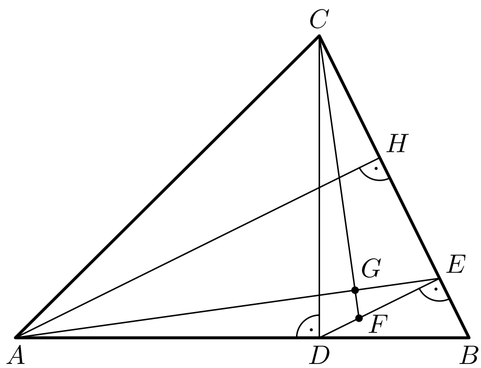

Fig. 4

Then the following equations are equivalent

$$\frac{EB}{HE} = \frac{DF}{EF},$$

$$\frac{BE}{EH} + 1 = \frac{DF}{EF} + 1,$$

$$\frac{BE + EH}{EH} = \frac{DF + FE}{EF},$$

and

$$\frac{BH}{EH} = \frac{DE}{FE}. \tag{8}$$

The right-angled triangles CDE and ABH are similar. Then $\angle ABH = \angle CDE$ holds, and

$$\frac{AH}{BH} = \frac{CE}{ED}. \tag{9}$$

Multiplying both sides of (8) by (9) we obtain

$$\frac{AH}{EH} = \frac{CE}{EF}.$$

From the above equality and similarity of the triangles CDE and ABH we can see that the triangles ECF and HAE are also similar. Then

$$\angle EGC = 180^\circ - (\angle ECG + \angle CEG)$$

$$= 180^\circ - (\angle HAE + \angle HEG) = 180^\circ - 90^\circ = 90^\circ,$$

so $CF \perp AE$.

Solution 5

Let H be the point of the segment BC such that DE is parallel to AH (see Fig. 4). Then the proportion

$$\frac{AD}{BD} = \frac{HE}{EB}$$

holds. Using (1) we obtain the proportion

$$\frac{AB}{BD} = \frac{EF}{DF}.$$

The straight line CF intersects the side DE in the same ratio as the straight line AE intersects the side BH. Thus the triangles ABH and CDE are similar. Then $\angle EAB = \angle FCD$ and we can calculate

$$\begin{aligned}
\angle AGC &= 180^\circ - \angle GAC - \angle GCA \\
&= 180^\circ - (\angle BAC - \angle BAE) - (\angle ACD + \angle FCD) \\
&= 180^\circ - \angle BAC + \angle BAE - (90^\circ - \angle BAC) - \angle FCD \\
&= 180^\circ - \angle BAC + \angle BAE - 90^\circ + \angle BAC - \angle FCD = 90^\circ,
\end{aligned}$$

so $CG \perp AG$.

Solution 6

Let P be a point belonging to the segment DE such that $CP \perp AE$. We will prove that $P = F$ (see Fig. 4 and Fig. 5).

We remark that

$$\angle EDC = 90^\circ - \angle BDE = \angle DBE.$$

Because three corresponding angles are equal, the triangles DEB and CED are similar. Then

$$\frac{BE}{BD} = \frac{DE}{DC}. \tag{10}$$

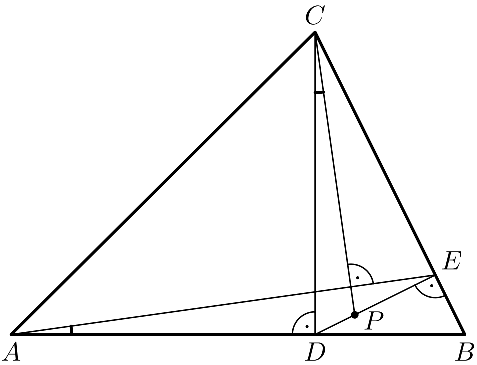

Fig. 5

From $CP \perp AE$ and $CD \perp AB$ we obtain $\angle EAB = \angle DCP$. Also $\angle EDC = \angle DBE$. Corresponding angles are equal and the triangles ABE and CDP are similar. Then

$$\frac{AB}{BE} = \frac{DC}{DP}. \tag{11}$$

Multiplying both sides of (10) and (11) we obtain

$$\frac{AB}{BD} = \frac{DE}{DP}.$$

Then the following equations are equivalent

$$\frac{AB}{BD} = \frac{DE}{DP},$$

$$\frac{AB}{BD} - 1 = \frac{DE}{DP} - 1,$$

$$\frac{AB - BD}{BD} = \frac{DE - DP}{DP},$$

$$\frac{AD}{BD} = \frac{EP}{PD}.$$

Using (1) we conclude $P = F$ and $AE \perp CF$.

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[1] Czeskie zadanie i jego 10 rozwiązań. *Matematyka* 5 (2004), 292–295.

[2] Maciocha, M.: Jedenaste rozwiązanie. *Matematyka* 1 (2005), 52.

[3] Kmiecik, P., Mitek, J.: Jeszcze trzy rozwiązania. *Matematyka* 5 (2005), 50–51.

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Problems from the Iberoamerican Mathematical Olympiads 2003, 2004 and 2005, with Comments and some Solutions

Francisco Bellot Rosado

Francisco Bellot Rosado, born in Madrid in 1941, has been chairman of Mathematics at the I.E.S. Emilio Ferrari from Valladolid since 1970. Involved in the preparation of olympiad students since 1988, he received the Paul Erdös Award from the WFNMC in 2000. He is Western Europe's representative of the WFNMC, and editor of the digital journal Revista Escolar de la O.I.M., *with more than 10000 subscribers as at May 2006.*

1 The Problems

XVIII Iberoamerican Mathematical Olympiad 2003

Mar del Plata, Argentina

Problem 18.1
We have two sequences, each one formed by 2003 consecutive integers, and a board of two rows and 2003 columns.

a) Determine (with reasons) if it is always possible to distribute the numbers from the first sequence in the first row, and those from the second sequence in the second row, in a such manner that the results obtained when the two numbers in each column are added, they form a new sequence of 2003 consecutive numbers.

b) What if there are 2004 integers instead 2003?

Problem 18.2
Let C and D be two points on a semicircle of diameter AB such that B and C lie in different halfplanes with respect to the line AD. Let M, N and P be the midpoints of AC, BD and CD, respectively. Let O_A and O_B be the circumcentres of triangles ACP and BDP, respectively. Prove that the lines O_AO_B and MN are parallel.

Problem 18.3
Paul is copying from the blackboard the following problem:

"Consider all the sequences of 2004 real numbers $(x_0, x_1, \ldots, x_{2003})$ *such that*

$$\begin{aligned} x_0 &= 1, \\ 0 \le x_1 &\le 2x_0, \\ 0 \le x_2 &\le 2x_1, \\ &\vdots \\ 0 \le x_{2003} &\le 2x_{2002}. \end{aligned}$$

Determine a sequence that the following expression S is at maximum . . . "

When Paul wishes copy the expression S, it hes been erased from the blackboard. He remembers only that S is in the form

$$S = \pm x_1 \pm x_2 \pm \cdots \pm x_{2002} + x_{2003},$$

where the last coefficient is $+1$, and all others are $+1$ or -1.

Prove that Paul can find the solution of the problem.

Problem 18.4
Let $M = \{1, 2, \ldots, 49\}$ be the set of the first 49 positive integers. Determine the biggest integer k such that M has a subset with k elements which does not contain 6 consecutive numbers. For this maximal value of k, find the number of k-subsets of M having such a property.

Problem 18.5
Let P and Q be points on the sides BC and CD, respectively, of the square $ABCD$, such that $BP = CQ$. Let us consider points X, Y $(X \neq Y)$, belonging to segments AP and AQ, respectively. Prove that for all X, Y there exists a triangle with sides of lengths BX, XY and YD.

Problem 18.6
We define sequences (a_n) and (b_n) in the following way:

$$\begin{aligned} a_0 &= 1; \qquad a_{n+1} = a_n^{2001} + b_n \\ b_0 &= 4; \qquad b_{n+1} = b_n^{2001} + a_n, \qquad n \geq 0. \end{aligned}$$

Prove that 2003 does not divide a_n nor b_n, for all n.

XIX Iberoamerican Mathematical Olympiad 2004

Castellón, Spain

Problem 19.1
Some squares of a 1001×1001 board are to be colored according to following rules:

(i) If two squares share a side, then at least one of them must be colored.

(ii) Among any six successive squares in a row or in a column some two adjacent ones must be both colored.

Determine the smallest number of squares that need to be colored.

Problem 19.2
In the plane are given a circle with center O and radius r and a point A outside the circle. For any point M on the circle, let N be the diametrically opposite point. Find the locus of the circumcentre of the triangle AMN when M describes the circle.

Problem 19.3
Let n and k be positive integers either with n odd or with both n and k

even. Show that there exist integers a and b such that

$$\gcd(a, n) = \gcd(b, n) = 1 \quad \text{and} \quad k = a + b.$$

Problem 19.4
Find all pairs (a, b) of two-digit natural numbers such that both $100a + b$ and $201a + b$ are four-digit perfect squares.

Problem 19.5
In a scalene triangle ABC, points A', B', C' are the intersection points of the internal bisectors of angles A, B, C with the opposite sides, respectively. Let BC meet the perpendicular bisector of AA' at A'', CA meet the perpendicular bisector of BB' at B'', and AB meet the perpendicular bisector of CC' at C''. Prove that A'', B'', C'' are collinear.

Problem 19.6
For a set H of points in a plane, we say that a point P of this plane is an *intersection point* if there exist distinct points A, B, C, D in H such that AB and CD intersect at P.

We are given a finite set A_0 of points in the plane. Let sets $\mathsf{A}_1, \mathsf{A}_2, \ldots$ be constructed inductively as follows: For every $j \geq 0$, A_{j+1} is the union of A_j and the set of intersection points of A_j.

Prove that if the union of all the sets A_j is finite, then $\mathsf{A}_j = \mathsf{A}_1$ for all $j \geq 1$.

XX Iberoamerican Mathematical Olympiad 2005
Cartagena de Indias, Columbia

Problem 20.1
Solve in the domain of real numbers the following system of equations

$$\begin{aligned} xyz &= 8, \\ x^2y + y^2z + z^2x &= 73, \\ x(y-z)^2 + y(z-x)^2 + z(x-y)^2 &= 98. \end{aligned}$$

Problem 20.2
A flea is jumping over the integer points of the number line. In his first move, it jumps from the point 0 to the point 1. Then, if in one move the flea jumps from point a to the point b, in the next move it will jump from point b to one of the points $b+(b-a)-1$ or $b+(b-a)+1$.

Prove that, if the flea falls down two times over the point positive integer n, then the flea at least has made t moves, in which $t = \lfloor 2\sqrt{n} \rfloor$.

Problem 20.3
Let p be a prime number ($p > 3$). If

$$\frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{(p-1)^p} = \frac{n}{m},$$

where the gcd $(n,m) = 1$, show that p^3 divides n.

Problem 20.4
Given two positive integers a and b, denote by $(a \nabla b)$ the residue obtained when a is divided by b. This residue is one of the numbers $0, 1, \ldots, b-1$.

Find all pairs (a,p) of positive integers such that p is prime and the equation

$$(a\nabla p) + (a\nabla 2p) + (a\nabla 3p) + (a\nabla 4p) = a + p.$$

holds

Problem 20.5
We are given an acute-angled triangle ABC in the plane with circumcentre O. The point A_1 belongs to the minor arc BC of the circumcircle of ABC. Let A_2, A_3 be points on the sides AB, AC, respectively, such that

$$B\widehat{A_1}A_2 = \widehat{OAC} \qquad \text{and} \qquad C\widehat{A_1}A_3 = \widehat{OAB}.$$

Prove that the orthocentre H of the triangle ABC belongs to the line A_2A_3.

Problem 20.6
We are given a positive integer n. In a plane consider $2n$ collinear points A_1, $A_2, \ldots,$ A_{2n}. Each point is colored in blue or red by means of the following method:

In the given plane, n circles are drawn, with diameters in the points A_i and A_j. The circles are two by two disjoints. Each A_k, $1 \leq k \leq 2n$, belongs exactly to one circle. The points are colored in a such way that the two points of the one circle are of the same colour.

Determine how many distinct colorings of the $2n$ points can be obtained when the n circles varies and also varies the distribution of the colours.

2 Links to see the results (i.e. the distribution of medals)

The following links show the distribution of medals in these Olympiads:

18 OIM: http://www.campus-oei.org/oim/xviiioimmedallas.htm

19 OIM: http://www.campus-oei.org/oim/xixoimmeda.htm

20 OIM: http://olimpia.uanarino.edu.co/xxoim/resultados.htm

3 The Puerto Rico cup

The Iberoamerican Mathematical Olympiad (OIM) is perhaps a unique competition in the world because it includes an explicit acknowledgement of the progress of a country in the successive contests. The so-called "Puerto Rico cup" is awarded to the country with the best progress in the Iberoamerican Olympiad. The procedure for awarding of the Puerto Rico cup is the following (translating from the Regulation of the Puerto Rico cup):

I For country to have the right to "fight" for the Puerto Rico cup in the Iberoamerican Olympiad in the year $x+2$, it must satisfy the two following conditions:

a) It must have taken part in the Iberoamerican Olympiads of the years x and $x+1$;

b) It must take part each year with a full team (4 students).

II The so-called "Jump of relative progress" S, reached by a country in the year $x+2$, is a number determined by the following procedure:

a) The sum of the points obtained by a country in the OIM's of the years x and $x+1$ will be divided by the total number of students who take part in the OIM in these years. This number will be multiplied by 100, and the result will be divided by the total average of the points of the years x and $x+1$. (This total average is, of course, the quotient between the sum of the points of all the students in both olympiads and the number of students of both olympiads). In this way we obtain the normalized average P of each country in the last two years.

b) The total number of points obtained by a country in the Olympiad of the year $x+2$ is divided by 4 and by the total average of points of this year, and it is multiplied by 100.

The number obtained Q is the normalized average for each country in this Olympiad.

c) Then the Jump of relative progress S equals

$$S = Q - \frac{11}{10}P.$$

The country which wins of the Puerto Rico cup is the one with the largest S score.

In the case of a tie between two or more countries, the following ordinated schema will be followed, always referring to the Olympiad $x+2$, by successive eliminations, until a winner is found.

a) The country with biggest Q
b) The country with more Gold medals
c) The country with more Silver medals
d) The country with more Bronze medals
e) The country with more Honourable mentions.

In 2003, the Puerto Rico cup was actually awarded to Puerto Rico, and in 2004 to Ecuador.

The name Puerto Rico cup is not a coincidence: in 1989 the then leader of the Puerto Rico team, Prof. José Luis Torres, who proposed in 1989 the creation of this award, with the commitment to annually award a cup to the winning country. The proposal and the Regulations of the cup were accepted unanimously by the International Jury of the IV OIM, held at the castle "de la Mota" in Medina del Campo, near Valladolid (Spain) in 1990. The last modification of the Regulations were adopted in 1999.

4 Solutions to two problems

Problem 20.1
Solve in the domain of real numbers the following system of equations

$$\begin{aligned} xyz &= 8, \\ x^2y + y^2z + z^2x &= 73, \\ x(y-z)^2 + y(z-x)^2 + z(x-y)^2 &= 98. \end{aligned}$$

Solution

Dividing the second equation by the first one, we obtain

$$\frac{x}{z} + \frac{y}{x} + \frac{z}{y} = \frac{73}{8}.$$

Substituting

$$u = \frac{x}{z}, \quad v = \frac{y}{x}, \quad w = \frac{z}{y},$$

we have

$$uvw = 1, \quad u + v + w = \frac{73}{8}.$$

Dividing the third equation of the given system by the first one we get

$$\frac{y}{z} + \frac{z}{y} - 2 + \frac{z}{x} + \frac{x}{z} - 2 + \frac{x}{y} + \frac{y}{x} - 2 = \frac{98}{8}$$

which using new unknowns can be written as

$$\frac{1}{u}+\frac{1}{v}+\frac{1}{w}=\frac{98}{8}+6-\frac{73}{8}=\frac{73}{8},$$

and further

$$uv+vw+wu=\frac{73}{8},$$

because $uvw=1$.

Hence u, v and w are the roots of the cubic equation

$$t^3-\frac{73}{8}t^2+\frac{73}{8}t-1=0 \Longleftrightarrow 8t^3-73t^2+73t-8=0,$$

which clearly have the real solution $t=1$. The other real solutions are the roots of the quadratic equation

$$8t^2-65t+8=0,$$

that is

$$t_1=8, \qquad t_2=\frac{1}{8}.$$

The values of (u,v,w) are all permutations of $\left(1,8,\frac{1}{8}\right)$. Going back to x, y, z in one of these permutations, we have

$$x=z, \qquad 8x=y, \qquad y=8z,$$

and as $xyz=1$, so

$$8x^3=8 \Longrightarrow x^3=1 \Longrightarrow x=1$$

and the solutions are of the form $(1,8,1)$ and all its permutations.

Problem 20.5

We are given an acute-angled triangle ABC in the plane with circumcentre O. The point A_1 belongs to the smallest arc BC of the circumcircle of ABC. Let A_2, A_3 points on the sides AB, AC, respectively, such that

$$\widehat{BA_1A_2}=\widehat{OAC} \qquad \text{and} \qquad \widehat{CA_1A_3}=\widehat{OAB}.$$

Prove that the orthocentre H of the triangle ABC belongs to the line A_2A_3.

Solution

In my solution to this problem I will use a relatively less known result, the theorem of the transversals, which establish necessary and sufficient conditions for line which cuts off two sides of a triangle passing by some of the notheworthy points of the triangle. See [1] for a detailed discussion.

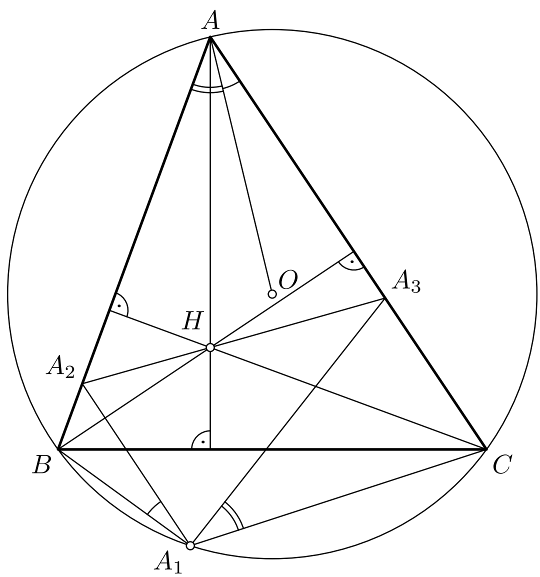

Fig. 1

As ABC is acute, then the necessary and sufficient condition in order that A_2A_3 passes by the orthocenter H is that

$$\frac{A_2B}{A_2A}\cdot\frac{\tan B}{\tan A}+\frac{A_3C}{A_3A}\cdot\frac{\tan C}{\tan A}=1. \tag{1}$$

We define $\beta=\widehat{CBA_1}$, $\gamma=\widehat{BCA_1}$.

First of all, we can observe that:

a) As AH and AO are isogonal lines in every triangle, we have

$$\widehat{OAC}\;=90^\circ-B,$$

$$\widehat{OAB} = A - (90^\circ - B) = 90^\circ - C.$$

b) As the quadrilateral ABA_1C is cyclic,

$$A = \beta + \gamma, \qquad \text{and also}$$
$$B + \beta \qquad \text{and} \qquad C + \gamma \qquad \text{are supplementary.}$$

Consequently we can compute $\dfrac{A_2B}{A_2A}$ and $\dfrac{A_3C}{A_3A}$. We will apply the sine rule at several triangles:

$$\frac{A_2B}{\cos B} = \frac{A_2A_1}{\sin(B+\beta)} \qquad \text{(sine rule in the triangle } A_2BA_1\text{)}$$

$$\frac{A_2A}{\cos A} = \frac{A_2A_1}{\sin\gamma} \qquad \text{(sine rule in the triangle } A_2A_1A\text{)}$$

from which

$$\frac{A_2B}{A_2A} = \frac{\sin\gamma}{\sin(B+\beta)} \cdot \frac{\cos B}{\cos A}. \tag{2}$$

By an analogous way, we obtain

$$\frac{A_3C}{A_3A} = \frac{\sin\beta}{\sin(C+\gamma)} \cdot \frac{\cos C}{\cos A}. \tag{3}$$

So, substituting (2) and (3) in the condition (1), we obtain after simplification,

$$\frac{\sin\gamma}{\sin(B+\beta)} \cdot \frac{\sin B}{\sin A} + \frac{\sin\beta}{\sin(C+\gamma)} \cdot \frac{\sin C}{\sin A} = 1.$$

We can write this condition in the equivalent form

$$\frac{BA_1}{AA_1} \cdot \frac{b}{a} + \frac{CA_1}{AA_1} \cdot \frac{c}{a} = 1,$$

using the fact that the triangles ABA_1, ACA_1, BCA_1 and ABC have the same circumcircle and applying to all of these the sine rule (extended, that is, for example $\dfrac{BA_1}{\sin\gamma} = 2R$).

But this equality is

$$BA_1 \cdot b + CA_1 \cdot c = a \cdot AA_1,$$

which is Ptolemy's theorem in the cyclic quadrilateral ABA_1C. The proof is complete.

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On One Property of an Integrable Function[1]

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1 Introduction

Mathematical competitions at various levels sometimes contain questions with rather paradoxical wording. For instance, in book [1, p. 105] we find:

Task 1
A man wrote down his incomes and costs every month. It is possible that his total costs for any five successive months exceeded his incomes whereas for the year in total his incomes exceeded his costs?

(In A. A. Yegorov's wording, the task sounds even more intriguing: "How long can a business partnership stay legal, if its operational results for any 5 months show profits but its annual report to the tax office demonstrates total absence of taxable profits ...")

The second example was offered at the 1977 XIX International Mathematical Olympiad in Beograd, see [2, p. 5]:

Task 2
In a finite sequence of real numbers the sum of any seven successive

members is negative, whereas the sum of any eleven successive members is positive. Find the largest quantity of numbers for such a sequence.

The papers of the 16th International Tournament of Towns, 1994–1995 (Autumn) [5, (6)] show the following task:

Task 3 (A. Kanel-Belov)
Two sequences have the periods of 7 and 13. What is the maximal length of their coinciding initial fragment? (A sequence period a_n is the smallest integer p such that the equality $a_n = a_{n+p}$ holds for any number n).

The mathematical idea for all problems of this kind can be formulated as:

For any sequence of linear real numbers the following conditions (1) are held:

1. A sum of any m successive members of S_m is negative;
2. A sum of any n successive members of S_n is positive. (1)

Determine maximum lenght $N_{\max}$ of this sequence.

The solution program for such problems consists generally of two parts:

1. The values of $N_{\max}$ should be indicated and substantiated for any n and m ;

2. An example of the task-satisfying sequence should be given, where the number of members equals $N_{\max}$.

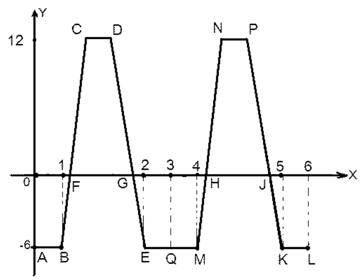
In our paper we will consider still another modification of the problem.

Task 4
Is there a continuous function $y = f(x)$ such that any definite integral of this function is negative for each interval of $m = 3$ length, but positive for each interval of $n = 5$ length?

Solution

Such functions do exist. Analytical expression for one such function (2) and its approximate graph scheme are given in the figure below. The graph is symmetrical with respect to the straight line $x = 3$. Each of the graph portions is also symmetrical: the first one with respect to the straight line $x = 1.5$, whereas the second (right-hand) portion is symmetrical with respect to the straight line $x = 5$. It also follows from (2) that for $x \in [0, 3]$, $f(x) = f(x + 3)$.

$$f(x) = \begin{cases} -6, & 0 \le x < 1 \\ 180x - 186, & 1 \le x < 1.1 \\ 12, & 1.1 \le x < 1.9 \\ -180x + 354, & 1.9 \le x < 2 \\ -6, & 2 \le x < 3 \\ -6, & 3 \le x < 4 \\ 180x - 726, & 4 \le x < 4.1 \\ 12, & 4.1 \le x < 4.9 \\ -180x + 894, & 4.9 \le x < 5 \\ -6, & 5 \le x \le 6 \end{cases} \tag{2}$$



Let us denote $X_3(3,0)$, $X_5(5,0)$, $X_6(6,0)$. After easy computation get the areas of the following five trapezoids: $OFBA$, $FCDG$, $GEMH$, $HNPJ$ and JX_6LK. These are

$$S_{OFBA} = \frac{61}{10},$$
$$S_{FCDG} = \frac{52}{5},$$
$$S_{GEMH} = \frac{61}{5},$$
$$S_{HNPJ} = \frac{52}{5},$$
$$S_{JX_6LK} = \frac{61}{10}.$$

It is easy to validate that the condition demand holds for this function: any definite integral of the function is negative for $m = 3$ interval, and positive for $n = 5$ interval.

In fact, let us examine the algebraic sum of squares for curvilinear trapezoids plotted above the (OX) axis and below it, from the interval of $m = 3$ length. If $x \in [0, 3]$, then we have:

$$S_{OFBA} + S_{FCDQ} + S_{GEQX_3} = -6.1 + 10.4 - 6.1 < 0.$$

As long as $f(x) = f(x + 3)$, for any $x \in [a, a + 3]$ and $0 < a \leq 3$, the algebraic sum of squares of curvilinear trapezoids plotted above the (OX) axis and below it from the interval of $m = 3$ length, will always remain negative. If now we consider the algebraic sum of squares of curvilinear trapezoids plotted above the (OX) axis and below it from the interval of $n = 5$ length, then we have for $x \in [0, 5]$:

$$\begin{gathered} S_{OFBA} + S_{FCDQ} + S_{GEQX_3} + S_{X_3QMH} + S_{HNPJ} + S_{JKX_5} = \\ = -6.1 + 10.4 - 6.1 + 10.4 - 0.1 > 0. \end{gathered}$$

By virtue of figure symmetry, a parallel shift for $\vec{r} = (b, 0)$, $0 \leq b \leq 1$ vector cannot change the ratio between the positive and the negative sums of curvilinear trapezoids.

The task is solved.

Let us proceed in our research.

We should prove the following theorem:

Theorem 1
The integrable function $f(x)$ in $[0, c]$ interval is known to possess the following features: any defined integral is positive in its n length segment, and is negative in its m length, with $c > m > n > 0$.

a) Prove that $c < m + n$;

b) Prove that if m and n are commensurable, i.e. $\frac{m}{n} = \frac{p}{q}$ is an irreducible fraction, p, q are integers, then $c < m + n - \frac{m}{q}$.

Proof

a) Based on proof by contradiction, let us assume that $c \geq m+n$. Let $F(x)$ be the antiderivative of $f(x)$ function. We denote the antiderivative function of $F(x)$ as $\Phi(x)$. Then, using the Newton-Leibniz formula and taking the problem condition into account (the positive function integral is positive, the negative function integral is negative), we have for $0 \leq x \leq c-m$, $0 \leq y \leq c-n$

$$0 < \int_0^m \left[\int_x^{x+n} f(y)\,\mathrm{d}y\right]\mathrm{d}x = \int_0^m (F(x+n)-F(x))\,\mathrm{d}x =$$
$$= \Phi(m+n) - \Phi(m) - \Phi(n) + \Phi(0) =$$
$$= \int_0^n (F(y+m)-F(y))\,\mathrm{d}y = \int_0^n \left[\int_y^{y+m} f(x)\,\mathrm{d}x\right]\mathrm{d}y < 0$$

or a contradiction. Thus, $c < m+n$.

b) Based on the commensurability condition, we have $d = \frac{m}{q} = \frac{n}{p}$, evidently $n = pd$ and $m = qd$. Let $c \geq m+n-\frac{m}{q} = (p+q-1)d$. For any $0 < d$, $k \in N$, $kd \leq c$, we denote $S_{d,k} = \int_{(k-1)d}^{kd} f(x)\,\mathrm{d}x$. Let us examine the sequence (I), comprised of $(p+q-1)$ members:

$$S_{d,1}, S_{d,2}, \ldots, S_{d,p+q-1}. \qquad (I)$$

According to the problem condition, this sequence satisfies the following properties:

1. A sum of any successive p members is positive;

2. A sum of any successive q members is negative.

We will prove and apply the following lemma:

Lemma 1
The following conditions (similar to conditions (1)) are hold for the sequence of line-written real numbers:

1. A sum of any m successive members of S_m is negative;

2. A sum of any n successive members of S_n is positive.

Then, the number of a given sequence members will be no more than $N_{\max} = m + n - d - 1$, where $d = (m, n)$ is the greatest common divisor of n and m.

Proof of Lemma 1

Without loss of generality it can be assumed that $m > n$. It is evident that if the problem has a solution, then n and m cannot be multiples of each other. Then the greatest common divisor of n and m equals $d = (n, m) \neq n$. Then $n > d$. Denote $n = n_1 d$, $m = m_1 d$; $(m_1, n_1) = 1$.

We can show that $N_{\max}$ cannot be increased and will use the *reductio ad absurdum* proof. We assume that the sequence has $N_1 = m + n - d = d(n_1 + m_1 - 1)$ members.

1. We break the sequence of N_1 length into $(m_1 + n_1 - 1)$ groups, having d numbers in each group. According to the problem condition, we have that the sum of numbers in any n_1 groups $S_n = S_{n_1 d} > 0$, and the sum of numbers in any m_1 groups $S_m = S_{m_1 d} < 0$.

2. We examine any $(m_1 - n_1)$ groups of successive members in our sequence. The remaining groups, containing the sequence members, are $(m_1 + n_1 - 1) - (m_1 - n_1) = 2(n_1 - 1)$. Since $2(n_1 - 1)$ is an odd number, then by any partition of the number into two summands, one of them should be no less than n_1. It means that $(m_1 - n_1)$ groups of the sequence members we have chosen at random can always be complemented from the left (or from the right wherever possible) by additional n_1 groups of numbers.

3. Now we complement our chosen $(m_1 - n_1)$ groups of members by n_1 groups of numbers. Let S_{m-n} be the sum of numbers in chosen $(m_1 - n_1)$ groups. It follows from $S_{n_1 d} = S_n > 0$ and $S_{m_1 d} = S_m < 0$ conditions that $S_{(m1-n_1)d} < 0$, which means that the sum of numbers in any chosen $(m_1 - n_1)$ groups is negative.

Let us redefine the problem condition:

For the $N_1 = m + n - d = d(m_1 + n_1 - 1)$ sequence of line-written real numbers partitioned into $(m_1 + n_1 - 1)$ groups, each having d numbers, the following conditions are fulfilled:

- the sum of members in n_1 predetermined S_n groups is positive;
- the sum of members in any predetermined $(m_1 - n_1)$ groups of $S_{(m_1-n_1)d} < 0$.

Note that the replacement of the number of m_1 groups by the number of $(m_1 - n_1)$ groups corresponds to the Euclidean algorithm step. We will repeat this argument as many times as necessary according to the Euclidean algorithm. Since $(m_1, n_1) = 1$, we have that the sum of any d numbers in a group must either be always positive or always negative. Because $m = m_1 d$ and $n = n_1 d$, we have that S_m and S_n are of the same sign. It is a contradiction, and thus, Lemma 1 is proved. ∎

According to Lemma 1, the sequence (I) can contain no more than $p + q - (p, q) - 1$ members, where (p, q) is the greatest common divisor of p and q. The contradiction obtained concludes the theorem proof. ∎

Remark 1
Thus, we see that the mathematical idea described in this paper's beginning can take diversified forms. At the same time, transition to the "integral topics" brings into view the conditions of incommensurable integration limits.

We revert to the proven theorem and proceed with our reasoning for incommensurable m and n.

Let us choose prime number p, assuming $d = \frac{n}{p}$, $q = \left[\frac{m}{d}\right]$ (where $[x]$ denotes the integer part of a number x). Then $n = pd$ and $m = qd - r$, where $0 < r < d$. Denote $m' = qd$.

We will plot a stepwise function $f(x)$ for p, q, d parameters at the $[0, m' + n - 2d]$ interval, corresponding to the "paradoxical" sequence, i.e. the sequence that satisfies the following properties:

1. The sum of any one of its successive p members is positive;

2. The sum of any one of its successive q members is negative.

(The algorithm for "paradoxical" sequence construction can be found in [4]).

As this takes place, a stepwise function accepts the value similar to the corresponding sequence member at the corresponding segment d. It means that for the function $f(x)$ at the $[0, m' + n - 2d]$ interval, any definite integral is positive in n length segment, and negative in m' length segment.

Denote $A = \max|f(x)|$, $M = \max \int_z^{z+m'} f(x)\,\mathrm{d}x < 0$.

It is now easy to show that if $0 < r < \frac{|M|}{A}$, then for the function $f(x)$ at $[0, m' + n - 2d]$ interval any definite integral for m length segment is also negative.

In fact,

$$\int_z^{z+m} f(x)\,\mathrm{d}x = \int_z^{z+m'} f(x)\,\mathrm{d}x - \int_{z+m}^{z+m'} f(x)\,\mathrm{d}x \leq$$
$$\leq M + |f(x)| \cdot (m' - m) \leq M + Ar < M + |M| = 0.$$

i.e. we can plot any function satisfying the condition for incommensurable m and n at the $[0, m + r + n - 2d]$ interval. Choosing a sufficiently large value of prime number p, we can obtain the examples of functions for incommensurable m and n at the interval, differing from $[0, m + n]$ by any minor value.

Our research is brought to completion by the next theorem.

Theorem 2

1. Prove that if m and n are commensurable, i.e. $\frac{m}{n} = \frac{q}{p}$ is an incontractible fraction and p, q are integers, then for any $\epsilon > 0$, $m + n - \frac{m}{q} > \epsilon$, an integrable function $f(x)$ can be constructed at $[0, m + n - \frac{m}{q} - \epsilon]$ interval such that any definite integral is

positive by n length segment, and negative by m length segment, $m > n > 0$.

2. Prove that if m and n are incommensurable, then for any $\epsilon > 0$, $m + n > \epsilon$, an integrable function $f(x)$ can be constructed at the $[0, m + n - \epsilon]$ interval such that any definite integral is positive by n length segment, and negative by m length segment, $m > n > 0$.

Proof

1. Assume that $d = \frac{m}{q} = \frac{n}{q}$, evidently, $n = pd$ and $m = qd$. Choose integer N such that $0 < \frac{d}{N} < \epsilon$. We examine integers $m_1 = Nq$ and $n_1 = Np$; evidently, $N = \gcd(m_1, n_1)$. We construct a stepwise function $y = f_1(x)$ defined at the $[0, m_1 + n_1 - N - 1]$ interval and corresponding to such "paradoxical" sequence that:

- The sum of all its successive m_1 members is positive;
- The sum of all its successive n_1 members is negative.

Such a function should definitely exist.

Let us examine the $y = f(x) = f_1\left(d\frac{x}{N}\right)$ function as a desired quantity. The function is defined on the

$$\left[0, \frac{m_1 + n_1 - N - 1}{N} d\right] = \left[0, m + n - \frac{m}{q} - \frac{d}{N}\right] \supset \left[0, m + n - \frac{m}{q} - \epsilon\right]$$

interval and satisfies the following properties:

- The sum of all its successive q members is positive;
- The sum of all its successive p members is negative.

Thus, the proof of the first part is finished.

2. Let us construct a chain of pairs of numbers (the construction principle is described in [4])

$$(m, n) \to (m_1, n_1) \to (m_2, n_2) \to \cdots \to (m_{l-1}, n_{l-1}) \to (m_l, n_l) \to \ldots$$

Because m and n are incommensurable, the chain will be infinite and there is always a number such that all pairs of figures following this number are less than any preset figure. For instance, we can take that $m_l + n_l < \epsilon$.

Assume that we have already constructed on the $[0, m_l + n_l - \delta_l]$ interval a function such that any definite integral is positive for the n_l length interval, and is negative for the m_l length interval.

Considering that the (m_i, n_i) pair can be obtained from the (m_{i-1}, n_{i-1}) pair by two ways: either $m_i = m_{i-1} - n_{i-1}$ or $n_i = n_{i-1} - m_{i-1}$, the function at the $[0, m_l + n_l - \delta_l]$ interval is constructed similarly to the "paradoxical sequence" construction. Namely, we will further define the function, depending on a specific case, either at the $[-m_{l-1}, 0]$ interval or at the $[-n_{l-1}, 0]$ interval and so that either $f(-x) = f(m_{l-1} - x)$ or respectively, $f(-x) = f(n_{l-1} - x)$, and then "shift" the coordinate origin to the left.

We will repeat such constructions until we obtain the $[0, m + n - \delta_l]$-defined function. Note that $0 < \delta_l < m_l + n_l < \epsilon$.

Let us prove the following lemma:

Lemma 2
Assume that the $[0, m_l + n_l - \delta_l]$-defined function meets the following condition: the meanings of all definite integrals for n_l length interval are equal to V_l, whereas the meanings of all definite integrals for m_l length interval are equal to W_l.

Then, for the function constructed on the $[0, m_{l-1} + n_{l-1} - \delta_l]$ interval according to above-mentioned algorithm the following property is met: the meanings of all definite integrals for the n_{l-1} length interval are equal to a certain V_{l-1}, whereas the meanings of all definite integrals for m_{l-1} length interval are equal to W_{l-1}. In this process, if pair (W_l, V_l) corresponds to pair (m_l, n_l) then for $(m_{l-1}, n_{l-1}) = (m_l + n_l, n_l)$ pair we have the correspondence of $(W_{l-1}, V_{l-1}) = (W_l + V_l, V_l)$ pair and when $(m_{l-1}, n_{l-1}) = (m_l, m_l + n_l)$, we have the correspondence of $(W_{l-1}, V_{l-1}) = (W_l, W_l + V_l)$.

Proof of Lemma 2

As it follows from the construction for $(m_{l-1}, n_{l-1}) = (m_l + n_l, n_l)$ pair, we will further define the $f(-x) = f(n_{l-1} - x)$ function at the $[-n_{l-1}, 0]$ interval, and then shift the origin of coordinates. It is evident that at the $[0, m_{l-1} + n_{l-1} - \delta_l]$ interval the meanings of all definite integrals for n_l length interval are equal to V_l, whereas the meanings of all definite integrals for $m_i + n_l$ interval are equal to $V_l + W_l$.

The case when $(m_{l-1}, n_{l-1}) = (m_l, m_l + n_l)$, is treated similarly. Thus, Lemma 2 is proved. ■

Using Lemma 2 and moving chainwise at the beginning, we have that the $(W_0, V_0) = (kW_l + pV_l, rW_l + tV_l)$ pair corresponds to the (m, n) pair.

It remains to prove now:

1. To construct a function at the $[0, m_l + n_l - \delta_l]$ interval where any definite integral for n_l length interval is equal to V_l, and for m_l length interval is equal to W_l.
2. To choose such coefficients that $kW_l + pV_l > 0$, $rW_l + tV_l < 0$

It is evident the (m_l, n_l) pair can be chosen so that it would satisfy the $n_l < m_l < 2n_l$ condition. Assume that $\delta_l = n_l$ and define the stepwise function at the $[0, m_l]$ interval as follows

$$y = f(x) = \begin{cases} A, & x \in [m_l - n_l, n_l), \\ B, & x \in [0, m_l - n_l) \cup [n_l, m_l]. \end{cases}$$

Then it is evident that any definite interval for n_l length interval is equal to $V_l = B(m_l - n_l) + A(2n_l - m_l)$, and for m_l length interval, to $W_l = A(2n_l - m_l) + 2B(m_l - n_l)$, i.e. we have the conditions when lemma is fulfilled.

We select the A and B parameters in such a manner that $kW_l + pV_l > 0$ holds. It can always be done. ■

The constructed stepwise integrable function can easily be transformed into the continuous one that satisfies the similar conditions.

Remark 2 (Summary of results)
The following statements follow from the mentioned proofs:

1. Let α, β be commensurable. Then there exists maximal $\delta > 0$ such that $\alpha = k_1\delta$, $\beta = k_2\delta$, $k_1 \in N$, $k_2 \in N$, $(k_1, k_2) = 1$. In this case, no paradoxical example can be constructed with $\gamma \geq (k_1+k_2-1)\delta$. With $\gamma < (k_1+k_2-1)\delta$, a paradoxical example can be constructed.

2. Let α, β be incommensurable, then the paradoxical situation is possible when $\gamma < \alpha + \beta$. With $\gamma \geq \alpha + \beta$, a paradoxical example cannot be constructed.

By examining the case of non-strict inequalities we have the following results:

3. Let α, β be incommensurable. If there exists a continuous function $f(x)$, defined at the $\gamma \geq \alpha + \beta$ length interval, whose integral is non-positive for any α length interval, and non-negative for any β length interval, then $f(x) \equiv 0$.

4. Let α, β be commensurable and $\alpha = k_1\delta$, $\beta = k_2\delta$, $k_1 \in N$, $k_2 \in N$, $(k_1, k_2) = 1$. If $\gamma \geq (k_1+k_2-1)\delta$, then an integral for any δ interval is equal to zero. In this case, the function is periodical with δ period.

Finally, we would like to offer a research problem:

5. In what Φ_1, Φ_2, Φ_3 regions there is a continuous function f with Φ_3 domain such that its integral for any parallel transfer of Φ_1 figure is positive, whereas its integral for any parallel transfer of Φ_2 figure is negative? Examine the case when Φ_1, Φ_2 figures are a rectangle and a circle.

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Tournament of Towns Corner

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Andrei Storozhev is a Research Officer at the Australian Mathematics Trust. He gained his Ph.D. at Moscow State University specializing in combinatorial group theory. He is a member of the Australian Mathematics Competition Problems Committee, Australian Mathematics Olympiad Committee and one of the editors of Mathematics Contests—The Australian Scene *journal.*

1 Selected Problems from the Second Round of Tournament 27

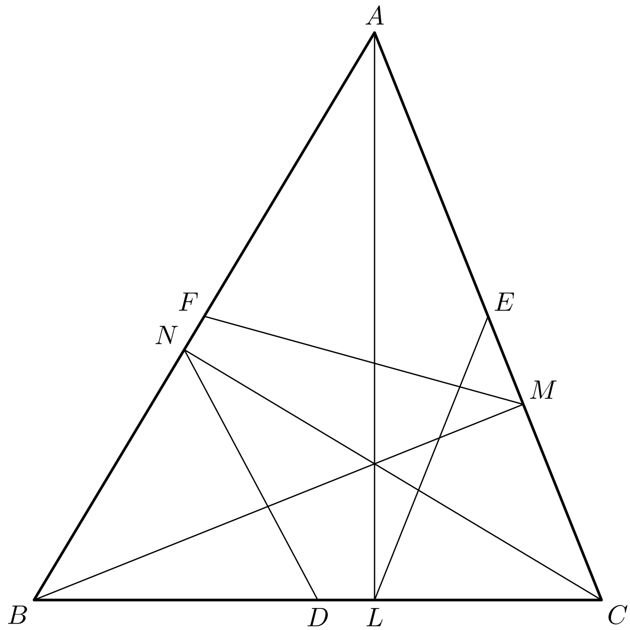
In the first round of Tournament 27, both Junior and Senior O Level papers consisted of five problems, while both Junior and Senior A Level papers were made up of six problems. Here are selected questions with solutions from the first round of Tournament 27.

1. One of 6 coins is a fake. We do not know the weight of either a real coin or of the fake coin, except that they are not the same. Using a scale which shows the total weight of the coins being weighed, how can the fake coin be found in 3 weighings?

Solution. Let the coins be A, B, C, D, E and F. In three weighings, we determine the average weight m of C and E, the average weight n of D and F, and the average weight k of B, E and F. If $m = n = k$, the fake coin is A. If $m = n \neq k$, the fake coin is B. If $m \neq n = k$, the fake coin is C. If $k = m \neq n$, the fake coin is D. If $k \neq m \neq n \neq k$, then the fake coin is E or F. This can be distinguished since $2m + n = 3k$ if it is E, and $m + 2n = 3k$ if it is F.

2. In triangle ABC, points D, E and F are the midpoints of BC, CA and AB respectively, while points L, M and N are the feet of the altitudes from A, B and C respectively. Prove that one can construct a triangle with the segments DN, EL and FM.

Solution. D is the midpoint of the hypotenuse of the right triangle NBC. Hence $DN = \frac{1}{2}BC$. Similarly, $EL = \frac{1}{2}CA$ and $FM = \frac{1}{2}AB$. Hence the segments DN, EL and FM can form a triangle half the linear dimensions of triangle ABC.



3. Initially, there is a rook on each of the 64 squares of an 8×8 chessboard. Two rooks attack each other if they are in the same row or column, and there are no other rooks directly in between. In each move, one may take away any rook which attacks an odd number of other rooks still on the chessboard. What is the maximum number of rooks that can be removed?

Solution. First, note that none of the corner rooks may be removed since each always attacks two other rooks. Moreover, we cannot leave behind only the four corner rooks, as otherwise the last to be taken away will attack two or zero other rooks. We can take away as many as $64 - 4 - 1 = 59$ rooks as shown in the diagram below.

•	1	2	3	4	5	6	•
13	50	51	52	53	54	55	7
14	15	16	17	18	19	56	8
20	21	22	23	24	25	57	9
26	27	28	29	30	31	58	10
32	33	34	35	36	37	59	11
38	39	40	41	42	43	•	12
•	44	45	46	47	48	49	•

4. On a circle are a finite number of red points. Each is labelled with a positive number less than or equal to 1. The circle is to be divided into three arcs so that each red point is in exactly one of them. The sum of the labels of all red points in each arc is computed. This is taken to be 0 if the arc contains no red points. Prove that it is always possible to find a division for which the sums on any two arcs will differ by at most 1.

Solution. For any arc A, denote by $f(A)$ the sum of the labels of the red points on A. Since there are finitely many red points, there are finitely many ways to divide them among three arcs. For each division, let the arcs be L, M and S, with $f(L) \geq f(M) \geq f(S)$. Choose among the divisions the one in which $f(L) - f(S)$ is minimum. We claim that for this division, $f(L) - f(S) \leq 1$. Suppose that this is not so. Now L and S are adjacent to each other. Let R be the red point on L closest to S and let r be its label. Consider the division (L', M', S') where the only change is that R moves from L to S. If $f(L) - r > f(S) + r$, then $f(L') = \max\{f(M), f(L) - r\}$ while $f(S') = \min\{f(M), f(S) + r\}$. On the other hand, if $f(L) - r \leq f(S) + r$, then $f(L') = \max\{f(M), f(S) + r\}$ while $f(S') = \min\{f(M), f(L) - r\}$. We have $f(L') - f(S') < f(L) - f(S)$ since $r \leq 1$, and $f(M)$ cannot be equal to $f(L)$ and $f(S)$ simultaneously. However, this contradicts our minimality assumption.

5. In triangle ABC, $\angle A = 2\angle B = 4\angle C$. Their bisectors meet the opposite sides at D, E and F respectively. Prove that $DE = DF$.

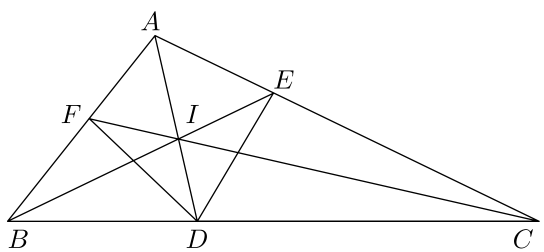
Solution. Let I be the incentre of triangle ABC. Let $\angle BCI = \angle ACI = \theta$. Then $\angle ABI = \angle CBI = 2\theta$ and $\angle CAI = \angle BAI = 4\theta$. Hence $\angle AIE = \angle BID = \angle BDI = 6\theta$, $\angle AIF = \angle AFI = 5\theta$ and $\angle AEI = 4\theta$. Let $AI = x$ and $DI = y$. Then $AF = IE = x$ and $BD = BI = x + y$. In triangle BAD,

$$\frac{AB}{AI} = \frac{DB}{DI}, \quad \text{so that} \quad BF = \left(\frac{DB}{DI} - \frac{AF}{AI}\right)AI = \frac{x^2}{y}.$$

In triangle ABE,

$$\frac{EA}{EI} = \frac{BA}{BI}, \quad \text{so that} \quad AE = \left(\frac{AF + FB}{BI}\right)EI = \frac{x^2}{y} = BF.$$

It follows that triangles EAD and FBD are congruent to each other, so that $DE = DF$.



2 World Wide Web

Information on the Tournament, how to enter it, and its rules are on the World Wide Web. Information on the Tournament can be obtained from the Australian Mathematics Trust web site at

http://www.amt.edu.au

3 Books on Tournament Problems

There are five books available on problems of the Tournament. Information on how to order these books may be found in the Trust's advertisement elsewhere in this journal, or directly via the Trust's web page.

Please note the Tournament's postal address in Moscow:

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To attain an appropriate level of achievement in mathematics, students require talent in combination with commitment and self-discipline. The following books have been published by the AMT to provide a guide for mathematically dedicated students and teachers.

Australian Mathematics Competition (AMC) Solutions and Statistics
Edited by DG Pederson

This book provides, each year, a record of the AMC questions and solutions, and details of medallists and prize winners. It also provides a unique source of information for teachers and students alike, with items such as levels of Australian response rates and analyses including discriminatory powers and difficulty factors.

Australian Mathematics Competition Book 1 (1978-1984)

Australian Mathematics Competition Book 2 (1985-1991)

Australian Mathematics Competition Book 3 (1992-1998)

Book 3 also available on CD (for PCs only).

Australian Mathematics Competition Book 4 (1999-2005)

Excellent training and learning resources, each of these extremely popular and useful books contains over 750 past AMC questions, answers and full solutions. The questions are grouped into topics and ranked in order of difficulty.

Problem Solving Via the AMC
Edited by Warren Atkins

This 210 page book consists of a development of techniques for solving approximately 150 problems that have been set in the Australian Mathematics Competition. These problems have been selected from topics such as Geometry, Motion, Diophantine Equations and Counting Techniques.

Methods of Problem Solving, Book 1
Edited by JB Tabov, PJ Taylor

This introduces the student aspiring to Olympiad competition to particular mathematical problem solving techniques. The book contains formal treatments of methods which may be familiar or introduce the student to new, sometimes powerful techniques.

Methods of Problem Solving, Book 2
JB Tabov & PJ Taylor
After the success of Book 1, the authors have written Book 2 with the same format but five new topics. These are the Pigeon-Hole Principle, Discrete Optimisation, Homothety, the AM-GM Inequality and the Extremal Element Principle.

Mathematical Toolchest
Edited by AW Plank & N Williams

This 120 page book is intended for talented or interested secondary school students, who are keen to develop their mathematical knowledge and to acquire new skills. Most of the topics are enrichment material outside the normal school syllabus, and are accessible to enthusiastic year 10 students.

International Mathematics — Tournament of Towns (1980-1984)

International Mathematics — Tournament of Towns (1984-1989)

International Mathematics — Tournament of Towns (1989-1993)

International Mathematics — Tournament of Towns (1993-1997)

International Mathematics — Tournament of Towns (1997-2002)
Edited by PJ Taylor

The International Mathematics Tournament of Towns is a problem solving competition in which teams from different cities are handicapped according to the population of the city. Ranking only behind the International Mathematical Olympiad, this competition had its origins in Eastern Europe (as did the Olympiad) but is now open to cities throughout the world. Each book contains problems and solutions from past papers.

Challenge! 1991 – 1995
Edited by JB Henry, J Dowsey, A Edwards, L Mottershead, A Nakos, G Vardaro

The Mathematics Challenge for Young Australians attracts thousands of entries from Australian High Schools annually and involves solving six in depth problems over a 3 week period. In 1991-95, there were two versions – a Junior version for Year 7 and 8 students and an Intermediate version for Year 9 and 10 students. This book reproduces the problems from both versions which have been set over the first 5 years of the event, together with solutions and extension questions. It is a valuable resource book for the class room and the talented student.

USSR Mathematical Olympiads 1989 – 1992
Edited by AM Slinko

Arkadii Slinko, now at the University of Auckland, was one of the leading figures of the USSR Mathematical Olympiad Committee during the last years before democratisation. This book brings together the problems and solutions of the last four years of the All-Union Mathematics Olympiads. Not only are the problems and solutions highly expository but the book is worth reading alone for the fascinating history of mathematics competitions to be found in the introduction.

Australian Mathematical Olympiads 1979 – 1995
H Lausch & PJ Taylor

This book is a complete collection of all Australian Mathematical Olympiad papers since the first competition in 1979. Solutions to all problems are included and in a number of cases alternative solutions are offered.

Chinese Mathematics Competitions and Olympiads 1981-1993 and 1993-2001
A Liu

These books contain the papers and solutions of two contests, the Chinese National High School Competition and the Chinese Mathematical Olympiad. China has an outstanding record in the IMO and these books contain the problems that were used in identifying the team candidates and selecting the Chinese teams. The problems are meticulously constructed, many with distinctive flavour. They come in all levels of difficulty, from the relatively basic to the most challenging.

Asian Pacific Mathematics Olympiads 1989-2000
H Lausch & C Bosch-Giral

With innovative regulations and procedures, the APMO has become a model for regional competitions around the world where costs and logistics are serious considerations. This 159 page book reports the first twelve years of this competition, including sections on its early history, problems, solutions and statistics.

Polish and Austrian Mathematical Olympiads 1981-1995
ME Kuczma & E Windischbacher

Poland and Austria hold some of the strongest traditions of Mathematical Olympiads in Europe even holding a joint Olympiad of high quality. This book contains some of the best problems from the national Olympiads. All problems have two or more independent solutions, indicating their richness as mathematical problems.

Seeking Solutions
JC Burns

Professor John Burns, formerly Professor of Mathematics at the Royal Military College, Duntroon and Foundation Member of the Australian Mathematical Olympiad Committee, solves the problems of the 1988, 1989 and 1990 International Mathematical Olympiads. Unlike other books in which only complete solutions are given, John Burns describes the complete thought processes he went through when solving the problems from scratch. Written in an inimitable and sensitive style, this book is a must for a student planning on developing the ability to solve advanced mathematics problems.

101 Problems in Algebra from the Training of the USA IMO Team
Edited by T Andreescu & Z Feng

This book contains one hundred and one highly rated problems used in training and testing the USA International Mathematical Olympiad team. These problems are carefully graded, ranging from quite accessible towards quite challenging. The problems have been well developed and are highly recommended to any student aspiring to participate at National or International Mathematical Olympiads.

Hungary Israel Mathematics Competition
S Gueron

This 181 page book summarizes the first 12 years of the competition (1990 to 2001) and includes the problems and complete solutions. The book is directed at mathematics lovers, problem solving enthusiasts and students who wish to improve their competition skills. No special or advanced knowledge is required beyond that of the typical IMO contestant and the book includes a glossary explaining the terms and theorems which are not standard that have been used in the book.

Bulgarian Mathematics Competition 1992-2001
BJ Lazarov, JB Tabov, PJ Taylor, AM Storozhev

The Bulgarian Mathematics Competition has become one of the most difficult and interesting competitions in the world. It is unique in structure, combining mathematics and informatics problems in a multi-choice format. This book covers the first ten years of the competition complete with answers and solutions. Students of average ability and with an interest in the subject should be able to access this book and find a challenge.

Mathematical Contests – Australian Scene
Edited by AM Storozhev, JB Henry & DC Hunt

These books provide an annual record of the Australian Mathematical Olympiad Committee's identification, testing and selection procedures for the Australian team at each International Mathematical Olympiad. The books consist of the questions, solutions, results and statistics for: Australian Intermediate Mathematics Olympiad (formerly AMOC Intermediate Olympiad), AMOC Senior Mathematics Contest, Australian Mathematics Olympiad, Asian-Pacific Mathematics Olympiad, International Mathematical Olympiad, and Maths Challenge Stage of the Mathematical Challenge for Young Australians.

WFNMC — Mathematics Competitions
Edited by Jaroslav Švrček

This is the journal of the World Federation of National Mathematics Competitions (WFNMC). With two issues each of approximately 80-100 pages per year, it consists of articles on all kinds of mathematics competitions from around the world.

Parabola incorporating Function

This Journal is published in association with the School of Mathematics, University of New South Wales. It includes articles on applied mathematics, mathematical modelling, statistics, and pure mathematics that can contribute to the teaching and learning of mathematics at the senior secondary school level. The Journal's readership consists of mathematics students, teachers and researchers with interests in promoting excellence in senior secondary school mathematics education.

ENRICHMENT STUDENT NOTES

The Enrichment Stage of the Mathematics Challenge for Young Australians (sponsored by the Dept of Education, Science and Training) contains formal course work as part of a structured, in-school program. The Student Notes are supplied to students enrolled in the program along with other materials provided to their teacher. We are making these Notes available as a text book to interested parties for whom the program is not available.

Newton Enrichment Student Notes
JB Henry
Recommended for mathematics students of about Year 5 and 6 as extension material. Topics include polyominoes, arithmetricks, polyhedra, patterns and divisibility.

Dirichlet Enrichment Student Notes
JB Henry
This series has chapters on some problem solving techniques, tessellations, base five arithmetic, pattern seeking, rates and number theory. It is designed for students in Years 6 or 7.

Euler Enrichment Student Notes
MW Evans and JB Henry
Recommended for mathematics students of about Year 7 as extension material. Topics include elementary number theory and geometry, counting, pigeonhole principle.

Gauss Enrichment Student Notes
MW Evans, JB Henry and AM Storozhev
Recommended for mathematics students of about Year 8 as extension material. Topics include Pythagoras theorem, Diaphantine equations, counting, congruences.

Noether Enrichment Student Notes
AM Storozhev
Recommended for mathematics students of about Year 9 as extension material. Topics include number theory, sequences, inequalities, circle geometry.

Pólya Enrichment Student Notes
G Ball, K Hamann and AM Storozhev
Recommended for mathematics students of about Year 10 as extension material. Topics include polynomials, algebra, inequalities and geometry.

T-SHIRTS

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Carl Friedrich Gauss T–shirt

The Carl Friedrich Gauss t-shirt celebrates Gauss' discovery of the construction of a 17-gon by straight edge and compass, depicted by a brightly coloured cartoon.

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George Pólya was one of the most significant mathematicians of the 20th century, both as a researcher, where he made many significant discoveries, and as a teacher and inspiration to others. This t-shirt features one of Pólya's most famous theorems, the Necklace Theorem, which he discovered while working on mathematical aspects of chemical structure.

Peter Gustav Lejeune Dirichlet T–shirt

Dirichlet formulated the Pigeonhole Principle, often known as Dirichlet's Principle, which states: "If there are p pigeons placed in h holes and p>h then there must be at least one pigeonhole containing at least 2 pigeons." The t-shirt has a bright cartoon representation of this principle.

Alan Mathison Turing T-shirt

The Alan Mathison Turing t-shirt depicts a colourful design representing Turing's computing machines which were the first computers.

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