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The aims of the Federation are:–

1. ***to promote excellence in, and research associated with, mathematics education through the use of school mathematics competitions;***
2. ***to promote meetings and conferences where persons interested in mathematics contests can exchange and develop ideas for use in their countries;***
3. ***to provide opportunities for the exchanging of information for mathematics education through published material, notably through the Journal of the Federation;***
4. ***to recognize through the WFNMC Awards system persons who have made notable contributions to mathematics education through mathematical challenge around the world;***
5. ***to organize assistance provided by countries with developed systems for competitions for countries attempting to develop competitions;***
6. ***to promote mathematics and to encourage young mathematicians.***

From the President

The Journal

In this issue, the WFNMC journal contains articles that keep us up to date on developments in competitions focusing on the Internet Mathematical Olympiad for University Students, the Middle European Mathematical Olympiad and the Tournament of Towns, articles containing new treatment of mathematical ideas such as the angle bisector theorem and the binary algorithm, and a look at novel problems for young students. This wealth of resources provided by the journal refreshes the perspectives of those of us dedicated to the creation and solving of challenging and original problems that delight and form young mathematical minds.

ICME-12

In just a little more than year it will be time for ICME-12 in Seoul; all involved in mathematics competitions and other challenges have time to reflect on the role WFNMC has to play there and prepare to offer Congress attendees a range of informative and inspiring talks that will certainly enrich and may redirect their professional lives.

How can WFNMC enrich and redirect the professional lives of mathematics teachers and educators working in the field of challenging mathematics itself?

There are three spaces that WFNMC and its members will have to enrich and focus work in the field of challenging mathematics; the first is the WFNMC miniconference to be held on 7 July, the day before ICME-12 begins, the second are the meetings of WFNMC that will be held during the Congress, and thirdly there are the contributions that WFNMC members and *all of our readers* can make to the proceedings of Topic Study Group 34, *The role of mathematics competitions and other challenging contexts in the teaching and learning of mathematics.* The aim is to provide different scenarios that will enable us to generate not only an exchange of perspectives, but experiences that will allow participants to mold, hone and reshape the ideas driving the work they have been doing in challenging mathematics. Strong initiatives that have prospered in one environment may also thrive in another when properly adapted to fit local conditions and aspirations.

Each Topic Study Group has been allotted four 90-minute sessions at ICME-12. The organizing team of TSG 34 aims to promote true dialogue and exchange of ideas by devoting two of those sessions to challenging mathematics beyond the classroom and one to working with teachers. I would like to see all of the reporting on competitions in one of these sessions, research and activities (programs) beyond the classroom different from competitons to be addressed in another, and all of the reporting on activities (programs) preparing teachers to introduce more challenging mathematics in their classrooms to be reported in a third session.

We are looking to organize one joint session between TSG 34 and TSG 3 *(Activities and Programs for Gifted Students)*. Our aim is that all contributions relating to the curriculum and activities in the classroom, both for challenging mathematics and for working with gifted students, be treated in the joint session we have proposed. This should provide for a well-rounded treatment of the curriculum, a point that is indispensable for the objective of providing mathematics in school, for all students as well as gifted students, that is challenging and that develops students' creativity.

One of the two WFNMC sessions will be devoted to awards and other important business. I believe the other should complement the topic study group session dedicated to reporting on research and new activities (programs) in the area of competitions.

At the miniconference, we aim to have two national exhibitions (by invitation) relating to challenging mathematics (both in and beyond the classroom, both competitions and other challenging activities and environments) that would be open for browsing at midday, during the breaks, and after the final afternoon session. There will be four sessions, two in the morning and two in the afternoon, each of 90 minutes, devoted to (1) problem creation; (2) the development of students' ability to think mathematically; (3) the development of teachers ability to think mathematically and creatively and of their capacity to create and/or use challenging problems in their teaching; and (4) resources for the student and the teacher (journals, books, webpages, etc.).

We invite all our readers to prepare their contributions; your contributions will not only contribute to the consolidation of work in the field of challenging mathematics itself, but furthermore with help to put forth a program that will attract and enrich the professional lives of teachers attending ICME 12 as well as math educators from other fields and specialties.

Is there a space for WFNMC to contribute to furthering and giving new/ renewed focus to the work of other math educators?

A brief look at questions raised by several topic study groups shows clearly that our work in challenging mathematics can profoundly enrich the exchange of ideas in areas of interest to many different topic study groups.

For example, the TSG on *number systems and arithmetic* has been focusing on the development of teaching/learning units that connect basic arithmetic skills with higher order thinking skills. The latter is exactly the aim of challenging problem solving on all levels, and WFNMC and its members provide leadership that will aid mathematics educators in this field to rethink the kinds of higher order thinking skills that young students can master and ways to entice students to commit to developing them.

Furthermore, one of the interests expressed by the TSG devoted to *algebra* concerns classroom processes that enhance the learning of algebra. Well-calibrated challenges are a time tested means of so doing since they provide strong motivation and involve the student as protagonist in the learning process. This shared interest can be forwarded by making contributions to the algebra group.

The TSG on *geometry* has been studying explanation, argumentation and proof in geometry education. There can be no doubt that one of the central interests in mathematical challenges are problems in geometry that require argumentation and proof, ranging from problems that elicit the construction of meaning for concepts such as area in primary school, thus contributing to the development of explanation, to those that are essentially new theorems constantly reinforcing the status of elementary geometry as an open system, to problems that enrich the possibilities of the geometry curriculum by showing the ways in which young students can master geometric transformations and use techniques lost to school geometry in creative ways.

The TSG concentrating on *discrete mathematics* has been addressing the questions of why and how to introduce discrete mathematics in schools. Activities in challenging mathematics have structured a corpus of topics in discrete mathematics that can be fruitfully proposed and developed with school children. Contributions from WFNMC and its members can supplement the work of this group.

The TSG dedicated to *reasoning, proof and proving* has been addressing the issue of the status of *reasoning, proof and proving* in mathematics as an academic subject. This is one of the three nuclei of challenging mathematics, accompanying and complementing heuristics and creativity. There can be no doubt that challenging mathematical activities stimulate reasoning that is coherent and sometimes surprising, as well as making proof an essential component of problem solving and developing young students' ability to justify and explain their reasoning processes.

WFNMC and its members have much to contribute to the TSG concerning *problem solving*. This group has shown its adoption of the legacy of Paul Halmos when it states that problem solving is the heart of mathematics, an activity that provides students with opportunities to construct and experience the power of mathematics. Relationships with this group should be strengthened and tightened; it has been our contention for several years now that challenge can be calibrated to provide for virtually all students that precise combination of motivation, success and frustration that makes solving beautiful problems such a strong attractor for students.

The TSG on *visualization*, designed to promote scholarship on the topic of visualization in the teaching and learning of mathematics, has many natural interrelations with challenging mathematics. Certainly visual representations permit students to conquer problems that would be meaningless or far too difficult presented or solved in formal mathematical language. Several different currents of thought, from Singapore to Colombia, have stressed the power of visual representations in problem solving and other challenging contexts.

Looking at the work of the TSG devoted to *learning and cognition in* *mathematics*, there is proclaimed an interest in students' formation of mathematical conceptions, notions, strategies, and beliefs. Here the development of mathematical thinking in its different manifestations can be carefully documented by mathematicians and educators working with mathematical challenges and in particular problem-solving competitions that require full explanations from the student.

Finally, the work of the TSG concerned with *curriculum development* has taken the position that mathematics curriculum development is a practical enterprise, one that impacts every teacher and student. It is also a scholarly activity that is based on evidence of what we know about what is worthwhile to learn, how students learn, how teachers use curriculum materials, and the developmental trajectories of mathematics content. Every one of these components can be informed and enhanced by contributions from those members of WFNMC who have devoted their work to structuring a more challenging mathematics curriculum.

Publications

Finally we look forward to receiving profound and delightful papers from ICME-12 and the WFNMC miniconference to be published in the December 2012 issue of this journal.

María Falk de Losada
President of WFNMC
Bogotá, June 2011

From the Editor

Welcome to *Mathematics Competitions* Vol. 24, No. 1.

First of all I would like to thank again the Australian Mathematics Trust for continued support, without which each issue (note the new cover) of the journal could not be published, and in particular Heather Sommariva, Bernadette Webster and Pavel Calábek for their assistance in the preparation of this issue.

Submission of articles:
The journal *Mathematics Competitions* is interested in receiving articles dealing with mathematics competitions, not only at national and international level, but also at regional and primary school level. There are many readers in different countries interested in these different levels of competitions.

- The journal traditionally contains many different kinds of articles, including reports, analyses of competition problems and the presentation of interesting mathematics arising from competition problems. Potential authors are encouraged to submit articles of all kinds.
- To maintain and improve the quality of the journal and its usefulness to those involved in mathematics competitions, all articles are subject to review and comment by one or more competent referees. The precise criteria used will depend on the type of article, but can be summarised by saying that an article accepted must be correct and appropriate, the content accurate and interesting, and, where the focus is mathematical, the mathematics fresh and well presented. This editorial and refereeing process is designed to help improve those articles which deserve to be published.

At the outset, the most important thing is that if you have anything to contribute on any aspect of mathematics competitions at any level, local, regional or national, we would welcome your contribution.

Articles should be submitted in English, with a black and white photograph and a short profile of the author. Alternatively, the article can be submitted on an IBM PC compatible disk or a Macintosh disk. We prefere LaTeX or TeX format of contributions, but any text file will be helpful.

Articles, and correspondence, can also be forwarded to the editor by mail to

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July 2011

The Internet Mathematical Olympiad for University Students and Some Thoughts on the Role of Competitions in the General Context of Mathematical Education

Alexander Domoshnitsky & Roman Yavich

Professor Alexander Domoshnitsky is Dean of Natural Sciences Faculty in Ariel University Center. His main specialization is the theory of differential equations. He is the author of more than 80 papers based on his own results in this area. His second specialization is mathematical education. He was an initiator of math classes in Israel schools and scientific camps for children. He is the author of the idea and organizer of Internet Mathematical Olympiad for students.

Dr. Roman Yavich specializes in informatics. He deals with the use of internet technologies in the education process. He published many papers in this area. He is the author of concept of technical support in organizing of Internet Mathematical Olympiad.

Over 100000 students and professionals have visited the International Internet Mathematical Olympiad site http://www.i-olymp.net since 2008. Thousands of students from as far as Brazil and the United States in the west and Vietnam in the east entered last year. Among the winners were champions of national and international mathematical competitions from Russia, Ukraine, Romania, Armenia, Brazil, Georgia and Israel.

1 Just a game?

At present dozens of competitions are held in different academic subjects. In mathematics, there are university, regional, national and international competitions; geometry competitions and algebra competitions... Anyone who has experience in the field and who can find a sponsor can organize a competition. And this is a good thing, because such competitions are undoubtedly beneficial for students. But the positive effect of these competitions on mathematical education in general—for instance, motivating students in a provincial college to improve their achievements—is very limited. Let us consider mathematical competitions from a different perspective, i.e. as a component in the general context of mathematical education. What should be the goals of mathematical competitions, based on the objectives of mathematical education?

We, educators who teach tech students in the periphery, devote most of our time and efforts to the weakest students, or, if we are lucky, students with average achievements. We strive to do everything we can in order to help these students, who constitute the majority in our colleges, achieve satisfactory results. But which of our students are going to occupy the most important positions in the technical and computer industries? The stronger students, whom the teachers do not have the time or the strength to tutor individually. And even more importantly, we have yet to come up with successful strategies for encouraging such students. Different strategies have been tried—special classes, student scientific societies, lectures on advanced material of the math courses. We have succeeded in some cases, but the absolute majority of students have shown very little interest in these projects. And unfortunately, this general lack of interest both on the part of the students and on the part of the teachers and the educational establishment, represented by deans and rectors, has doomed most of such endeavors.

It appears that only one strategy of motivating strong students has, though with varying levels of success, survived for almost 150 years.

It originated in Romania in the 1860s and it is called "mathematical olympiads". What is the secret of this phenomenon? If we examine the matter more closely, it becomes apparent that this secret lies in turning a very serious process—testing the students' level of knowledge and their ability to think creatively when solving mathematical problems—into a GAME that has certain simple rules and can therefore be easily understood both by participants and by observers. Perhaps, understanding this can enable us to fulfil our goals. By developing the game aspect, we can achieve an increase in the number of participants, and thus generate more interest in mathematical competitions both among potential participants and among observers.

2 Games and their Admirers

"An audience at a mathematical competition—that's ridiculous!" some skeptics might say. But allow us to disagree. Let us compare mathematical competitions to... chess. Chess hardly appears to be an exciting sport to watch, and yet those of us who belong to the older generation remember the national interest in the World Chess Championship games. During the Karpov-Kasparov matches, every evening on the national news the chess commentator, a grandmaster, would analyze the competitors' positions, and millions of chess fans (and not just they!) would copy these positions on their chessboards, in order to better understand the grandmaster's explanations and to try and figure out right then, that day—and not the next day, when the final part of the game would be played—who was going to win. As a result, you could see dozens of chessboards at the beaches and in parks and courtyards, and thousands of children and adolescents joined the chess elite each year. So let us ask ourselves, why should another intellectual game, one which has the potential to benefit large masses of college and high school students (and not just a select few), be unable to draw even minimal attention from professionals and potential observers?

"Who is the current World Chess Champion?" I ask my friends, those who once used to follow the chess championships so closely. They don't know. The widespread national popularity of the game has declined to the point that the majority of the population has lost all interest in it. This example is very instructive, since the main reason for the loss of popular interest in the game was the destruction of the established form of the competition, which viewers were familiar with and could easily understand. People stopped watching the games, and national interest in chess disappeared. Form and content. We learned about this in philosophy courses and we remember that form reflects content, and sometimes even influences content... And when the system includes an active subject (in this context, high school and college mathematics teachers), who formulates the objectives and changes the form in order to make the goal achievable, I believe it is definitely possible to awaken the students' and teachers' interest.

3 Objectives and Format

Let us first formulate our objectives, since the format that we choose for our mathematical competition depends mainly on them. If, for instance, our main objective were to teach students how to effectively prove their mathematical constructs, and also help them learn to listen attentively, detect logical holes or lack of proof, and find contrary examples, then we could not choose a better format than the one that was developed in mathematical schools in Moscow and Leningrad during the Soviet period, called "Mathematical Battle". But this competition was intended for the future mathematical elite, while our competition is for tech students, who will use mathematics in the future and who need to be able to understand what area a mathematical problem belongs to and what mathematical constructs they can build. Our objective is to awaken an interest in mathematics in as many students as possible, to teach them to fully understand mathematical problems and to think creatively, not just solve problems by using the existing methods that have been developed by others.

When organizing a mathematical competition, one must formulate specific methodological objectives. The rules of the game depend on the goal. Suppose, for instance, that the goal were selecting 6-8 students for the national team that will compete in the international Olympiad. In this case the difficulty level should be very high. Our goal, however, is very different, and since we want as many students as possible to participate, our olympiad should be built in a completely different way. The main guiding principle in pedagogy is, "Do no harm". Total failure in a competition can cause a good student, who is however not particularly gifted in mathematics, to feel psychologically traumatized and to lose confidence in his abilities. This student will be very unlikely to participate in the next olympiad. Thus, the number of participants will decline and we will be left with only the 5-6 students who have a real chance of winning.

A dialogue between an average or average-plus student who has failed completely and his parents would probably go along the lines of:

"How many problems did you solve?"

"None."

"Then you shouldn't participate. You're only shaming yourself!"

This is not what we want. Our goal is to create a competition which can interest both the average students and the best students in the country. However, if the questions are too easy, they will hold no interest for the most gifted students. And if the best students don't compete, the olympiad becomes simply another test, nothing more than that. Therefore, we have decided that the list of problems for the olympiad should be relatively long, and that it should include both simple and difficult problems. This way, the dialogue between the average-plus student and his parents will sound very different:

"How many problems did you solve?"

"Four."

"And the winner?"

"Six."

The difference between the number of problems that this student solved and the number of problems solved by the winner may be the same in both cases—two, but now the average student can also feel proud of his achievements.

Our list of problems includes 6 relatively easy and 6 relatively difficult problems. The high level of difficulty of some of these problems is evidenced by the fact that no one has yet solved all of the problems on our list, even though the participants included winners of other national and international competitions. Obviously, with this kind of problem list, determining the winner based simply on the number of problems solved is very problematic. Therefore, we allocate points based on each problem's rating. Let us suppose, for instance, that a certain problem is "worth" a thousand points. If a thousand students solve it, then each receives one point, but if only two solve it, each receives 500. Thus, you cannot receive enough points to become one of the winners by solving only the standard problems. In order to be among the best, you have to solve some of the most difficult problems. And besides, this rating system also gives talented high school students or students from provincial colleges a chance to win by solving a problem which requires creative mathematical thinking but doesn't require advanced mathematical knowledge.

4 Utilizing the Internet

"It takes two to tango"—and, similarly, in order to hold a mathematical competition, you need to have at least several contestants. At times it can be difficult to find enough potential contestants in one class or even in one university, and that is one of the reasons the International Internet Mathematical Olympiad, which enables people from all over the world to compete with each other, was started.

The Internet Mathematical Olympiad has a very simple format: at the appointed hour the conditions of the problems appear on our internet site, and the students, who are seated in front of computers in their universities, are given 4 hours to solve the problems and send us their answers via email. Video and audio connection allows the students to ask the organizers questions about the conditions of the problems during the competition. In 2006 we held the first Internet Olympiad for Israeli students. It was successful beyond expectations. To the surprise of the organizers, the participants included not only future technicians, computer programmers, physicists and mathematicians, but also students from other faculties, including future nurses. So, why shouldn't we continue trying to strengthen these students' interest in mathematics, helping them improve their mathematical skills and encouraging them to continue solving non-standard mathematical problems?

We soon found like-minded educators in other countries as well, and the second Internet Olympiad, which was held in 2007, included participants from Russia, Ukraine, Romania, Bulgaria and Germany.

One of the main problems for teachers who want their students to participate in an international mathematical competition has always been raising enough funds for the trip to a faraway city or country, funds which are then transferred to travel agencies, instead of being spent on the students' training. Internet Olympiads eliminate this problem, and therefore, all of the funds donated by sponsors can be given as a grant to teachers who train students for the Olympiad, or used as an additional incentive for students. The possibility of receiving a stipend from a sponsor can of course strengthen the students' motivation.

5 The Difference between the Internet Olympiad and the Mathematical Kangaroo

The format of the Internet Olympiad is somewhat similar to that of the famous Mathematical Kangaroo—an international mathematical competition which also utilizes the internet, but resembles a regular test. One of the main differences between the two, however, is that in the Internet Olympiad the students have to write a full solution of the problems, not just answer a multiple-choice question, and that makes it possible for a participant to receive points for an elegant or original solution, even if he has made a mistake along the way. This cannot be achieved in a multiple-choice test.

An additional difference is the joint opening ceremony and closing ceremony, which are held on the internet using the video connection. So many people have taken part in these ceremonies. Over the years we have seen students and professors from Moscow State University of Economics, Statistics, and Informatics (MESI), Moscow State Institute of Radiotechnics, Electronics and Automation (MIREA), Baltic State Technical University "Voenmeh", Yerevan State University, Saratov State University, Nizshnekamsk University, the Alexander Cuza University of Cluj Napoca, Mogilev University in Belorussia, etc. The results are announced at the joint closing ceremony of the Olympiad. You can see on screen the triumphantly raised hands of the winners, their tears of joy and their first interviews. During the closing ceremony the team leaders and the heads of universities also get a chance to speak. There is always an air of excitement in the conference room of Ariel University Center during these days. Students and teachers sit together and watch the students from other countries who are competing with them. You feel that you belong to some global club with no borders. And isn't that in itself worth something?

6 Who are the judges?

The choice of problems for the Olympiad is obviously a very important task. The problems must not only vary in their level of difficulty, but also be interesting in some way, require a certain level of ingenuity, and help the students understand some new mathematical facts or nuances. The main author of problems for the Internet Olympiad and the chairman of the jury is Prof. Aleksey Kannel-Belov from Bar-Ilan University in Israel, who is known both in Israel and in Russia as a mathematical composer, an author of a book about non-standard problems, one of the leading specialists in olympiad problems worldwide, the main coach of the Israeli national student team on International Mathematical Olympiads, and in the recent past the trainer of the national German student team. Another judge and author of problems is Dr. Vadim Bugaenko, who immigrated to Israel two years ago from Moscow. Dr. Bugaenko is also an author of a book about non-standard problems and was among the organizers of many mathematical competitions in Moscow. Today he is a staff member at Ariel University Center in Israel.

7 What's next?

In the future we intend to hold Olympiads combining several rounds of the Internet Olympiads, and after that a traditional final round. In November 2008 we invited a team from Moscow Radio University and a team from Yassi Technical University in Romania to come to Ariel. They met students from three Israeli universities during the special competitions and, of course, had an interesting excursion program in Israel.

We greatly value our collaboration with our main Russian partner, Prof. Vladimir Navodny from the Accreditation Agency of the Ministry of Education of Russia. In 2009 and 2010 the final game of our Internet Olympiad and the final of the Russian National Internet Olympiad were held together, and it was a great success for us both.

8 Problems of Internet Mathematical Olympiad

Final Round of the Season 2008–2009, May 14, 2009

1. Let A and B denote square matrices of the same order, and let $ABA = BAB$. Prove that one of the following conditions is satisfied: one of the matrices is degenerate, or A and B have equal determinants.

2. A pharmaceutical company is advertising a new product. Its representatives claim that using this product will result in a daily decrease in body weight or in cholesterol levels, or, during most days, both together. A customer used the new product regularly for a month, yet at the end of the month both his weight and his cholesterol levels were the same as before he started. Is it possible that the company's claims are true? Explain your answer.

3. Find all the solutions x_1, x_2, x_3, x_4, x_5 of the system

$$\begin{aligned} x_5 + x_2 &= yx_1, \\ x_1 + x_3 &= yx_2, \\ x_2 + x_4 &= yx_3, \\ x_3 + x_5 &= yx_4, \\ x_4 + x_1 &= yx_5, \end{aligned}$$

where y is a parameter.

4. Let $0 < a < b$. Find the area of the union of ellipses

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad \text{and} \qquad \frac{x^2}{b^2} + \frac{y^2}{a^2} = 1.$$

5. Is it possible to find 5 vectors in space such that all the angles between them are greater than $90°$?

6. Calculate $\int_1^{\mathrm{e}} \sqrt{\ln x}\,\mathrm{d}x + \int_0^1 \mathrm{e}^{x^2}\,\mathrm{d}x$.

7. The function $F(x)$ is continuous for all $x > 0$. It is known that for any fixed $x > 0$ the sequence $F(x+n) \to 0$, where $n \to \infty$. Does this mean that $\lim_{x\to\infty} F(x) = 0$?

8. A segment, the length of which is 1, is moving in such a way that both of its edges remain on the coordinate axis (in the first coordinate angle). As it moves along the plane, the segment changes the color of the part of the coordinate angle to its left. Find the equation of the line that separates the part of the plane the color of which has been changed from the part the color of which remains the same.

9. A ring has been placed on the end of a rod the length of which is 1 meter. At a certain point in time the rod begins to elongate uniformly (any two segments of equal length will be elongated equally during any given period of time). The ends of the pole are moving away from each other at the speed of 10 cm/sec. At the same point in time the ring begins to move towards the opposite end of the pole at the speed of 1 cm/sec. Will the ring ever reach the opposite end of the pole? If so, how much time will it take? (The width of the ring can be ignored).

10. Suppose that 2009 guests are present at a banquet hall. Each person knows at least 45 of the other guests. Prove that it is possible to find at least 4 guests who can be seated at a round table in such a way that each will be sitting next to a person whom he or she knows.

11. An infinite number of points is given. All of the distances between the points are integers. Prove that all of the points are situated on the same line.

12. A robot is searching for a certain tool. It moves along a plane, and each of its steps equals 1. It can move in any direction it chooses. When the distance between it and the tool is less than one step it can pick the tool up. After each step the robot knows whether the distance between him and the tool has decreased or increased.

Suppose that at a given point in time the distance between the robot and the tool equals N steps and the robot know the number of steps that separate him from the tool.

a) Prove that the robot can reach the tool in $[N + 10\log_2 N]$ steps (where $[x]$ denotes the integer part of x).

b) Prove that no algorithm exists which can enable the robot to reach the tool in less than $N + \frac{\log_2 N}{10}$ steps.

Problems from all previous Internet Olympiads and their solutions can be found on our website http://www.i-olymp.net .

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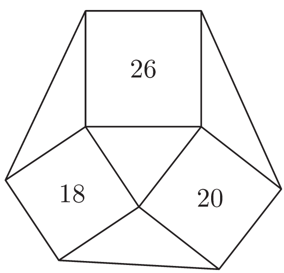
Some Problems from Training for a Junior Olympiad

Francisco Bellot-Rosado

Francisco Bellot Rosado, born in Madrid in 1941, has been chairman of Mathematics at the I.E.S. Emilio Ferrari in Valladolid since 1970. Involved in the preparation of Olympiad students since 1988, he received the Paul Erdös Award from the WFNMC in 2000. He is Western Europe's representative of the WFNMC, and editor of the digital journal Revista Escolar de la O.I.M.

We will present some problems useful in the training of the students who wish take part in a Junior Olympiad, from different sources which are quoted after each statement. We will start with an old but interesting problem from a book of 1917.

Problem 1
The picture shows three squares, of respective areas 26, 18 and 20 square units. Find the area of the hexagon constructed how the picture shows.

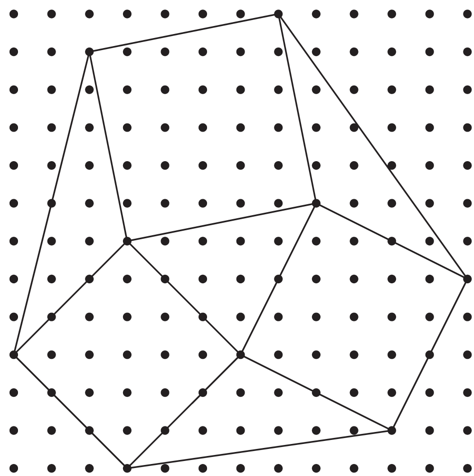


Source: "Amusements in Mathematics", by H. E. Dudeney, First edition 1917.

- The problem has an easy trigonometrical solution but there is a more interesting procedure to find a solution which be suitable for students without trigonometry knowledge.
- This is a versatile problem, in the sense that it is possible to present it to students from 12 years old up to 17 years old.

The key idea is the following:

- It is possible to express 18, 20 and 26 as a sum of two squares?
- $18 = 3^2 + 3^2$; $20 = 4^2 + 2^2$; $26 = 5^2 + 1^2$
- From this, the best is to see a picture below and to count small squares.

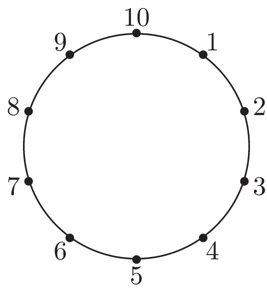


- The area of the hexagon is 100; each of the four triangles are equal to 9.
- The situation can be used as a way to discover Pick's theorem.

A variant of this problem was presented by the late Jim Totten in the Cariboo College of Kamloops (B.C., Canada): if the area of two of the squares equals the area of the rest of the figure, find the measure of one of the angles of the central triangle (the answer is 45°).

Problem 2
Ten people are seated around a round table. Each one thinks of one number and whispers it into the ear of the people seated on his right and left side. Then, every person says out loud the half of the sum of the numbers each one has heard.

The numbers each person says are shown in the picture.



Find the number thought of by the person who says "6".

Source: Hungarian School journal "KöMal" (Közepiskolai Matematikai Lapók)

- Let $x_1, x_2, \ldots, x_{10}$ be the numbers which are thought by the people who said $1, 2, \ldots, 10$ respectively. Immediately we write

$$\begin{aligned} x_1 + x_3 &= 4, & x_2 + x_4 &= 6,\\ x_3 + x_5 &= 8, & x_4 + x_6 &= 10,\\ x_5 + x_7 &= 12, & x_6 + x_8 &= 14,\\ x_7 + x_9 &= 16, & x_8 + x_{10} &= 18,\\ x_9 + x_1 &= 20, & x_{10} + x_2 &= 2. \end{aligned}$$

- If we sum all the equations on the right, we get

$$2(x_2 + x_4 + x_6 + x_8 + x_{10}) = 50$$

  And so $2(6+x_6+18) = 50$, that is, $x_6+24 = 25$ and finally $x_6 = 1$.

- Although the statement does not ask for it, it is possible to find all the numbers.

Problem 3
In a local contest of soccer, only four teams take part: A, B, C, D. Each team played exactly one match against each other. Team A beat team B by 4 goals to 1. Team D beat team C. All the other matches were drawn. The final classification, *by the total of goals scored*, was: 1st A; 2nd B; 3rd C and 4th D. The totals are all different. What was the total of goals in the match C *versus* D?

Source of the problem: Problem set of problems for the Kangaroo Math Contest 2010.

We can start the analysis by means of a table of the goals of each team:

	A	B	C	D
A		4	m	n
B	1		s	t
C	m	s		x
D	n	t	y	

The goals scored by the teams are: A: $4 + m + n$; B: $1 + s + t$; C: $m + s + x$; D: $n + t + y$

As these goal totals are in decreasing order, the sum of the goals of A and B exceed by at least 4 thee sum of goals of C and D. So, $5-(x+y)$ is at least 4, and this means $x+y = 0$ or $x+y = 1$. But $y > x$ and then $x = 0$, $y = 1$.

The goals of A exceed by at least 2 the goals of C, so $4 - (s - n)$ is at least 2, or equivalently, $s - n$ is not bigger than 2. Analogously, from the goals of B and D we conclude that $s - n$ is at least 2. Therefore, $s - n = 2$.

The number of goals in decreasing order are: $4+m+n$, $3+n+t$, $2+m+n$ and $1 + n + t$; from this we conclude $-1 < t - m < 1$ and so $t = m$.

The table can be rewritten as follows:

	A	B	C	D	Goals in favor	Goals received
A		4	m	n	$4+m+n$	$1+m+n$
B	1		$n+2$	m	$3+m+n$	$6+m+n$
C	m	$n+2$		0	$2+m+n$	$3+m+n$
D	n	m	1		$1+m+n$	$m+n$

So, team D beat team C by $1:0$.

Problem 4

Alex and Petar are at the Riga Station waiting for a train. To make the wait less boring, they decide to play in the following way when a freight train passes on the station, without stopping or reducing the speed. They are together on the platform; when the engine of the freight train reaches the point where they are standing, Alex starts to walk in the same direction of the train, and Petar in the opposite direction, both at the same speed. They stop the moment when the last wagon passes by the point on which they are at that moment. Alex walked 45 meters and Petar 30.

How long is the train?

Source of the problem: Junior Olympiad of Portugal, 2006.

Alex walked $15 = (45-30)$ meters more than Petar, and in the period of time in which Alex walked these 15 meters, the train advanced $45+30 = 75$ meters. Therefore, in the same period of time, the train advanced $75/15 = 5$ times further than each the protagonists. Then, while Petar walked 30 meters, the train advanced $30 \times 5 = 150\,\text{m}$. As Petar starts to walk when the train passes where he was, and stops when the last wagon passes, and walked $30\,\text{m}$ in the opposite way to the train, the train is $150 + 30 = 180\,\text{m}$ long.

Problem 5

Let a, b, c, d be integer numbers and n a positive integer. Suppose that:

1. n divides the sum $a+b+c+d$ and
2. n also divides the sum $a^2+b^2+c^2+d^2$.

Prove that n divides the sum $a^4 + b^4 + c^4 + d^4 + 4abcd$.

Source of the problem: Math. Summer Camp, Zakopane, Poland, presented by Prof. Lev Kurliandchuk, St. Petersburg University

Outline of the solution. The fact that in the statement appears some symmetrical functions of the numbers a, b, c, d suggests to consider the polynomial whose roots are just these numbers:

$$P(x) = (x - a)(x - b)(x - c)(x - d),$$

and using the development of the polynomial and the well-know fact that $P(a) = P(b) = P(c) = P(d) = 0$ we are directed to the solution.

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[3] Problem set of problems for the Kangaroo Math Contest 2010. Minsk (Belarus)

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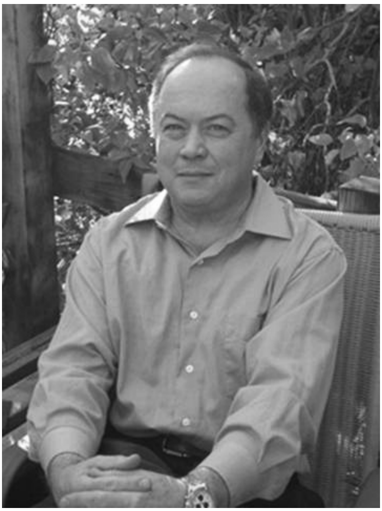
“In Order to Form a More Perfect Union ...”

Alexander Soifer



Born and educated in Moscow, Alexander Soifer has for 31 years been a Professor at the University of Colorado, teaching Mathematics, Art History and European Cinema courses. He has written 7 books and over 200 articles, including the following books published by Springer: The Mathematical Coloring Book *(2009)*, Mathematics as Problem Solving *(2009)*, How Does One Cut a Triangle? *(2009)*, Geometric Etudes in Combinatorial Mathematics *(2010)*, Ramsey Theory Yesterday, Today and Tomorrow *(2010, editor)*, The Colorado Mathematical Olympiad and Further Explorations *(2011). Soifer founded and for 28 years has run the Colorado Mathematical Olympiad. He served on the Soviet Union Math Olympiad (1970–1973) and USA Math Olympiad (1996–2005).*

I would like to present here another new example of a bridge between problems of mathematics and problems of mathematical Olympiads. Graph Theory has been a fertile ground for noticing beautiful ideas that would work very well in the Olympiad-type competitions. The problem presented here served as problem 4 in the April 23, 2010 Colorado Mathematical Olympiad.

In order to create this problem, I needed to know how many countries comprised United Nations. I guessed 200 – and was quite close!

Wikipedia informs: “The United Nations Organization (UNO) or simply United Nations (UN) is an international organization whose stated aims are facilitating cooperation in international law, international security, economic development, social progress, human rights, and the achieving of world peace... There are currently 192 member states, including nearly every sovereign state in the world."

And now the problem:

"In Order to Form a more Perfect Union ..." (A. Soifer)

The United Nations includes 192 member states, every pair of which has a disagreement. In order to form a more perfect Union, a *negotiation* is introduced: if representatives of four member states are seated at a round table so that each pair of neighbors has a disagreement, the negotiation resolves one of these four disagreements. A series of consecutive negotiations reduces the total number of disagreements to n. What is the *minimum* of n?

Solution

1) Let each member state be represented by a vertex of a graph, in which we connect two vertices by an edge if and only if the corresponding countries have a disagreement. Then the Initial Disagreements graph is the complete graph K_{192} on 192 vertices (a set of 192 vertices, every two of which are connected by an edge). A negotiation selects a 4-cycle C_4 of a graph ("representatives of four countries are seated at a round table so that each pair of neighbors has a disagreement") and removes one edge from it. The problem, translated into this language, asks to find the minimum number of edges in a Disagreements graph obtained from the initial K_{192} by a series of consecutive removals of an edge from a 4-cycle.

2) Observe first that the removal of an edge in a C_4 subgraph preserves connectivity of a graph (i.e. ability to travel between any pair of points through a series of edges).

If the series of consecutive negotiations were to eliminate all cycles, we would get a connected cycle-free graph, called a *tree*, on 192 vertices. Such a tree has exactly 191 edges (proof by an easy induction).

Observe that for any two points of a tree we have a unique path connecting them through a series of edges (for otherwise we would have created a cycle in the union of two distinct paths). This observation allows us to show that *any tree is 2-colorable* (so that vertices of the same color are not adjacent). Indeed, color a point A in color 0, and any other point B in color 0 or 1 depending upon the parity of the edge distance from A to B.

Observe now that the property of 2-colorability is preserved under the operation converse to negotiation, i.e. the operation of completing a 4-path to a 4-cycle. Therefore, if we assume that the graph with 191 edges is achieved, the Initial Disagreements graph is also 2-colorable. However, the Initial Disagreements graph K_{192} is not 2-colorable (it requires 192 colors!), therefore we will never get a tree as a result of a series of negotiations! We proved that 191 is unreachable.

3) On the other hand, we can fly a kite and in the process get a Disagreements graph with 192 edges.

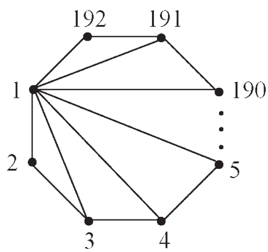


Figure 1: Kite-0

Through the series of negotiations we can get from the Kite-0 graph, which is K_{192}, to the Kite-1 graph, which consists of K_{191} with an attached 1-edge "tail." Indeed (see Figure 1), from the 4-cycle $\{1, 3, 4, 5\}$ we remove the edge $\{1, 3\}$; from $\{1, 4, 5, 6\}$ remove $\{1, 4\}$; $\ldots$, from $\{1, 190, 191, 192\}$ remove $\{1, 190\}$; from $\{1, 191, 192, 2\}$ remove $\{1, 191\}$. Finally, from the 4-cycle $\{1, 2, 3, 192\}$, we remove $\{1, 192\}$, getting the desired graph Kite-1 (Figure 2).

Continuing this process (you can formalize it by a simple induction), we will get to Kite-189 graph, which consists of K_3 with a tail of length 189

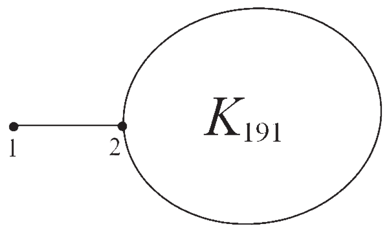


Figure 2: Kite-1

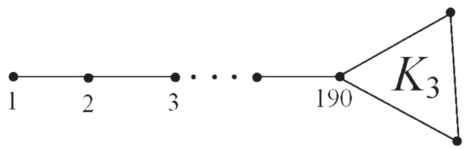


Figure 3: Kite-189

(Figure 3), which has exactly 192 edges as desired.

Homework for Children & Consenting to Homework Adults

Determine which of the graphs in Figures 4 and 5 can be obtained from the Initial Disagreements graph K_{192} through a series of negotiations.

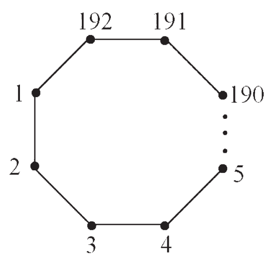


Figure 4

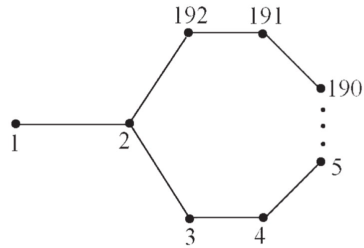


Figure 5

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A Prominent Correlation on the Extended Angle Bisector

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1 Introduction

In this short paper the author adduces a prominent correlation predicating or suggesting a new proposition with respect to the extension of the internal angle bisector up to an equal distance to the angle bisector, generating a particular quadrilateral which is consisted of diagonals emerging as angle bisectors and medians.

2 Proposition of the Problem

Let ABC be any triangle with the standard notation such that $AB = c$, $BC = a$, $CA = b$. Let D be a point on BC such that AD bisects the angle A and thereafter AD is extended up to P such that $AD = DP$ (see the picture below). Thus the quadrilateral $ABPC$ is formed with a peculiar correlation of its side lengths such that

$$\frac{p^2}{c} + \frac{q^2}{b} = b + c.$$

3 The Proof of the Proposition

AD is the internal angle bisector of angle BAC which cuts BC at D. AD is extended up to P as $AD = DP$. Moreover $AB = c$, $AC = b$, $BC = a$, $BP = p$, $PC = q$. A particular quadrilateral generated by the extended angle bisector CD is the median of ACP, hence using Apollonius' Theorem we have

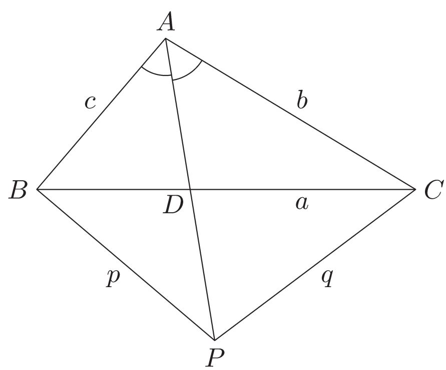
$$b^2 + q^2 = 2AD^2 + 2DC^2. \tag{1}$$

As AD is the bisector of angle A, it is well known that

$$\frac{BD}{DC} = \frac{c}{b}.$$

Therefore

$$BD = \frac{ac}{b+c} \qquad \text{and} \qquad DC = \frac{ab}{b+c}.$$



The standard length of the internal angle bisector of angle A can be adduced as

$$AD^2 = bc\left(1 - \frac{a^2}{(b+c)^2}\right).$$

(The author has already given a new proof for the standard length of the angle bisector in [3] in addition to the standard proof.)

Substituting these results to (1)

$$b^2 + q^2 = 2bc\left(1 - \frac{a^2}{(b+c)^2}\right) + 2\frac{a^2b^2}{(b+c)^2}$$

Therefore

$$\frac{a^2}{(b+c)^2} = \frac{b^2 + q^2 - 2bc}{2b(b-c)}. \tag{2}$$

BD is a median of ABP, hence using the Apollonius' Theorem again, we have

$$p^2 + c^2 = 2AD^2 + 2BD^2$$

$$p^2 + c^2 = 2bc\left(1 - \frac{a^2}{(b+c)^2}\right) + 2\frac{a^2c^2}{(b+c)^2} \tag{3}$$

By substituting for $\dfrac{a^2}{(b+c)^2}$ from (2), we obtain

$$p^2 + c^2 = 2bc\left(1 - \frac{b^2 + q^2 - 2bc}{2b(b-c)}\right) + 2c^2\frac{b^2 + q^2 - 2bc}{2b(b-c)},$$

$$(p^2 + c^2)b(b-c) = 2b^2c(b-c) + c(c-b)(b^2 + q^2 - 2bc),$$

$$p^2b + c^2b = c(b^2 - q^2 + 2bc).$$

After easy manipulation we get

$$\frac{p^2}{c} + \frac{q^2}{b} = b + c.$$

Remark. This correlation or the proposition can be used very widely throughout many prominent sections in mathematics, particularly in order to discover various new sporadic corollaries and propositions in Advanced Euclidean Geometry.

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Faster than the Fastest, or Can the Binary Algorithm Be Overhauled

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That which is in locomotion must arrive
at the half-way stage before it arrives at the goal.
Zenon of Elea. Aporia Dichotomy.

1 Introduction

The modern world cannot be conceived without calculators and computers. These computing means can promptly add, multiply, raise to a power, and perform numerous additional operations. However, fast raising to powers is reduced to fast multiplication, whereas fast multiplication is reduced to fast addition. Certain algorithms, promoting to accelerate the calculations, have been known from time immemorial.

In our paper, taking some tasks as an example, we want to compare different algorithms, and observe which of them operate faster, and in what cases. Some of these problems, despite their simple formulations, turned out to be hard nuts to crack in the general case. We will formulate the problems in the manner they have been given for the particular cases with corresponding reference to the source, and if the source did not contain a generalization, we will formulate our corresponding generalization for these problems as a separate point. We will formulate the problems that are sometimes very similar to each other and interrelated but have, nevertheless, certain differences. For those who do not want to rest on their laurels, we will offer a number of novel tasks for research and investigation.

2 Problems

Problem 1

a) In [1] is given: a box of granulated sugar, a pan balance and a 1-g plummet. How can 1 kg of sugar be weighed and given to a customer as fast as possible? (Show the equilibration pattern);

b) For those who are more accustomed to the non-metric system of weights and measures, solve the problem for a 1-ounce plummet and 100 pounds of sugar (1 pound = 12 ounces);

c) How can a generalized problem of weighing n grams of sugar using a pan balance and a 1-g plummet be solved?

Problem 2 see [2]
Only two operations are permitted to perform over a number: "to increase two-fold" and "to increase by 1". What is the smallest number of operations that permits to obtain: a) 100; b) n from 0, if the sum of binary notation for n is equal to s?

Problem[1] **3** see [3]
The value of x^8 can be found from the preset value of x in three arithmetic operations: $x^2 = x \cdot x$, $x^4 = x^2 \cdot x^2$, $x^8 = x^4 \cdot x^4$; the value of x^{15}, in five operations: the first three steps are the same, then come $x^8 \cdot x^8 = x^{16}$ and $x^{16} : x = x^{15}$. Prove that

a) x^{1000} can be found in 12 steps (multiplication and division);

b) for each positive integer n, x^n value can be found in no more than $\frac{3}{2}\log_2 n + 1$ operations.

Problem 4 see [4]
What is the smallest number of multiplications that is necessary for raising x to n power?

Problem 5

a1) Folklore: You have two glass marbles and a 100-storey building. You drop a marble from various floors as you want to know at what storey your marble breaks down from falling (for instance, it gets broken by falling from the 5^{th} floor, and still remains intact by falling from the 4^{th} floor). The question is: what minimal number of steps do you need to find out exactly at what floor the marbles start breaking down? (a marble can break down at any storey).

a2) The same question but with regard to a n-storey building.

b) Let us assume that the number of marbles is limitless. What minimal number of tests (drops from the building floors) is surely sufficient to determine a marble strength (i.e. the floor when the marbles start breaking down). The same question but with regard to a n-storey building.

Problem 6

[1] It is assumed in Problems 3 and 4 that $x \neq 0$ and $x \neq 1$.

a) In [5]: 199 trees of different ages are growing in a circle. Can the age of 12 trees be found out so that the tree, which is older that its both (right-hand and left-hand) neighbors, is defined most certainly?

b) n-trees of different ages are growing in a circle, $n \geq 3$. The age of what minimal number of trees should be found out in order to find the tree which is most certainly older than both (right-hand and left-hand) its neighbors?

3 Discussion of problems

In each of these problems, we can provide a relevant minimal number of operations (actions, weighs) for each natural n, thus obtaining a proper numerical sequence for each task. Actually, we have certain algorithms for each problem so that the sequence nth member is exactly equal to the number of steps of a corresponding algorithm. Thus, the examples of Problem 3 conditions display the algorithms for x^8 and x^{15} production, in 3 and 5 steps, respectively, and since these algorithms are minimal (check it), then the 8th member of the sequence is equal to 3, and the 15th member of the sequence is equal to 5.

Let us introduce denominations for the sequences we have:

$w(n)$ is a minimal number of weighs required for obtaining n grams of sugar at the pan balance with 1-g plummet. (Problem 1c));

$t(n)$ is a minimal number of operations required for obtaining n (Problem 2b);

$m(n)$ is a minimal number of actions (multiplication and division) required for raising x to n power (Problem 3);

$l(n)$ is a minimal number of multiplications required for raising x to n power (Problem 4);

$b(n)$ is a minimal number of tests in the marbles problem (Problem 5b)).

Let us tabulate some initial values of these sequences:

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$w(n)$	1	2	2	3	3	3	3	4	4	4	4	4	4	4	4	5
$t(n)$	1	2	3	3	4	4	5	4	5	5	6	5	6	6	7	5
$m(n)$	0	1	2	2	3	3	4	3	4	4	5	4	5	5	5	4
$l(n)$	0	1	2	2	3	3	4	3	4	4	5	4	5	5	5	4
$b(n)$	1	2	2	3	3	3	3	4	4	4	4	4	4	4	4	5

It can be seen from the table that the first sequence follows definite regularity that can be easily formulated—just take a closer look at the sequence members that occupy the 2nd, the 4th, the 8th, and the 16th place. These data are quite sufficient to formulate the induction proposition.

Also, the regularity is found for the second sequence, though it is not as noticeable as the first one. For instance, take a look at the values of those sequence members that occupy odd places, and the values of those sequence members that precede them. Though, definitely, some additional reasoning is necessary for the induction proposition.

As far as we have chance that the multiplication and division operations will assist us in achieving the desirable result faster than the multiplication operations only, then $m(k) \leq l(k)$. Strange as it may appear, we can see that the initial terms of $m(n)$ and $l(n)$ sequences coincide. Nevertheless, there is an infinitely large number of k values, where $m(k) < l(k)$. In these two cases, the exact formulas for the sequence members have not been found. However, the values of the sequence members, or as we named them, the number of steps of a corresponding algorithm, can be assessed.

So, what unifies all these problems? In our opinion, a binary technique or dichotomy[2] can be applied to all these problems. Probably, it is correct to say "a binary technique" when we double something, and use the term "dichotomy" when we divide something in two.

Let us examine clause b) of the first Problem. It is evident that we can weigh the same amount of sugar as the amount that is already found in a balance pan. Thus, using the plummet, we weigh 1 ounce of sugar, remove the plummet and weigh another ounce of sugar and bulk all sugar in one balance pan. So, we have 2 ounces of sugar in one balance pan,

[2]Dichotomy— Greek $\delta\iota\chi o\tau o\mu\acute{\iota}\alpha$: $\delta\breve{\iota}\chi\tilde{\eta}$, "in two" + $\tau o\mu\acute{\eta}$, "incision"

again weigh another 2 ounces of sugar, collect all sugar on one balance pan, and have 4 ounces of sugar already. We repeat this procedure, and, consequently, have 8, 16, 32, 64, 128, 256, 512, 1024 ounces of sugar. Since 100 pounds = 1200 ounces, and $1200 = 1024 + 128 + 32 + 16$, we can manage, using no more than $30 = 11 + 8 + 6 + 5$ weighs. And if we, having 16 ounces of sugar, would have weighed these 16 ounces of sugar once and transferred them to another container, then having weighed another 16 ounces of sugar, put them to the second balance pan, we can again have 32 ounces on the balance pan. Using the same procedure and transferring 32 ounces (and then 128 ounces) to another container, we would manage the case in $14 = 11 + 1 + 1 + 1$ weighs. It is clear that this weighing algorithm is rather fast though not minimal. The fact is that we removed the plummet after the first weighing and did not use it more.

Use of the plummet in our subsequent weighs permits us not only to avoid using an additional container, but to obtain minimal weighing algorithm as well. The subtlety is that we can place the plummet both on one balance pan and on the other one. Thus, if have weighed k ounces of sugar, then by subsequent weighing we can have $k - 1$, k or $k+1$ ounces of sugar on the other balance pan, depending on whether we employ the plummet or not, and if we do employ it, then what balance pan it is placed on. Having bulked the sugar, we can produce $2k - 1$, $2k$ or $2k + 1$ ounces of sugar at this stage, respectively. It is due to this subtlety that the values of $w(n)$ sequence members, starting from 2^q and until $2^{q+1} - 1$, are all equal to $q + 1$. Putting it differently,

$$w(n) = [\log_2 n] + 1,$$

where $[x]$ is the integer part of x. We leave to the reader's discretion the simple proof of this fact, using the mathematical induction technique. Since $2^{10} = 1024 < 1200 < 2047 = 2^{11} - 1$, minimal number of weighings for 100 pounds of sugar amounts to 11.

Let us examine the second problem in the light of the first one. It is easy to conceive that $t(2^q) = q + 1$. We can see that based on the condition, we can obtain either $k + 1$ or $2k$ out of k at the next step. Further similarity to the first problem appears to end here. But we can attempt to solve the problem. Let us proceed from the end and apply the operations opposite to those found in the conditions to the number obtained at a certain step, i.e. we will divide by 2 and subtract 1. We can easily construct the chain

$$100 \xrightarrow{1/2} 50 \xrightarrow{1/2} 25 \xrightarrow{-1} 24 \xrightarrow{1/2} 12 \xrightarrow{6} 6 \xrightarrow{1/2} 3 \xrightarrow{-1} 2 \xrightarrow{1/2} 1 \xrightarrow{-1} 0.$$

Now, we must reverse the operations again, and we will have the desired minimal chain

$$0 \xrightarrow{+1} 1 \xrightarrow{\times 2} 2 \xrightarrow{+1} 3 \xrightarrow{\times 2} 6 \xrightarrow{\times 2} 12 \xrightarrow{\times 2} 24 \xrightarrow{+1} 25 \xrightarrow{\times 2} 50 \xrightarrow{\times 2} 100. \qquad (*)$$

Here is our binary technique in operation!

It is evident that if we start operating at the end and obtain an odd number, we can only subtract 1. But if we obtain an even number, then by virtue of the fact that we are looking for a minimal chain, we must divide by 2, because if we subtract a unity, the number of paces will be higher. Strict demonstration of the fact can also by made by the mathematical induction technique.

In the second problem, however, we are concerned with the prompt that is mentioned in the problem conditions, namely the phrase "the sum of binary notation for n". The point is that not only binary notation of a number but the parameter itself (the sum of binary notation figures) plays one of the key roles both in this problem and in subsequent ones as well. But we will speak of it later.

In the first two problems we multiplied by 2, added or subtracted 1, while in the third and fourth problems we will raise our numbers to a power. Is there really any connection between these problems? The point is that we do not use the base number in any of the cases and perform all operations only with power exponents.

Thus, the instance of Problem 3 of x^8 value obtained from x is similar to obtaining 8 from 1 by doubling. Let us examine the above constructed chain $(*)$, starting from 1. We will have the following fast algorithm of raising to the 100th power:

$$x \cdot x = x^2, \quad x^2 \cdot x = x^3, \quad x^3 \cdot x^3 = x^6, \quad x^6 \cdot x^6 = x^{12},$$
$$x^{12} \cdot x^{12} = x^{24}, \quad x^{24} \cdot x = x^{25}, \quad x^{25} \cdot x^{25} = x^{50}, \quad x^{50} \cdot x^{50} = x^{100}.$$

It seems that except the "shift" (in Problem 2 the operations begin with 0, while in Problems 3 and 4 the operations begin with 1) we have the algorithm we need to solve Problem 4. To eliminate the "shift", we must only examine the sequence $t(n)-1$, as its corresponding algorithm has exactly one step less than the $t(n)$ sequence. For the initial values, number 1 should be subtracted from each line cell at the corresponding table line. To see the difference between the problems, let us check how x^{15} can be obtained from x.

The binary technique gives us

$$15 \xrightarrow{-1} 14 \xrightarrow{1/2} 7 \xrightarrow{-1} 6 \xrightarrow{1/2} 3 \xrightarrow{-1} 2 \xrightarrow{1/2} 1$$

and reversed

$$1 \xrightarrow{\times 2} 2 \xrightarrow{+1} 3 \xrightarrow{\times 2} 6 \xrightarrow{+1} 7 \xrightarrow{\times 2} 14 \xrightarrow{+1} 15,$$

6 steps in total.

In terms of raising to a power we have the following fast algorithm:

$$x \cdot x = x^2, x^2 \cdot x = x^3, x^3 \cdot x^3 = x^6, x^6 \cdot x = x^7, x^7 \cdot x^7 = x^{14}, x^{14} \cdot x = x^{15}.$$

But we can act differently:

In this way,

$$x \cdot x = x^2,\ x^2 \cdot x = x^3,\ x^3 \cdot x^3 = x^6,\ x^6 \cdot x^6 = x^{12},\ x^{12} \cdot x^3 = x^{15};$$

or in this way,

$$x \cdot x = x^2,\ x^2 \cdot x = x^3,\ x^3 \cdot x^2 = x^5,\ x^5 \cdot x^5 = x^{10},\ x^{10} \cdot x^5 = x^{15}.$$

These algorithms include 5 steps! Note that we have produced x^{15} by 5 steps and by multiplication only, without division (Cf. the instance of Problem 3). Consequently, the binary technique is a *fast* but not always *minimal* algorithm.

In the previous example we applied the so-called *method of multipliers* for fast raising to a power. Both short chains of our example have been constructed based on the fact that $3 \cdot 5 = 5 \cdot 3 = 15$. Recollecting the property of powers $x^{mk} = (x^m)^k = \left(x^k\right)^m$, the second algorithm of chain construction also becomes clear. If we have factorization with differing from 1 factors of $n = mk$, then we must first find x^m, and after that raise the result to power k. But if n is a prime integer, then we, using this technique, find x^{n-1} and multiply by x at the last step.

Which of the techniques is better depends on n. Both these algorithms can be "modernized", and, certainly, they can be combined. There is still another technique but we will discuss it later.

So far, the problem of finding a minimal algorithm for fast raising to a power has not yet been solved in the general case and may well be taken as a challenging problem. At the same time, various mathematicians have obtained the results that we would like to cite here, and that may be considered as independent problems and exercises because their solutions are well within the powers of schoolchildren and students.

As has already been said, the base number is not being used, thus, we can re-formulate Problems 3 and 4 in terms of addition and subtraction (multiplication by two is the same as addition of the same value to a number).

We offer the reformulated Problems 3 and 4 here.

1. calculator can only add the numbers stored in its memory. Initially, figure 1(unity) is stored at the calculator memory, and the amount of memory cells is unlimited. The question is: what minimal number of operations will allow obtaining n from 1?
2. calculator can only add or subtract the numbers stored in its memory. Initially, figure 1 (unity) is stored at the calculator memory, and the amount of memory cells is unlimited. The question is: what minimal number of operations will allow obtaining 1 from n?

But let us go back to the binary notation of n. Let

$$n = 2^q b_q + 2^{q-1} b_{q-1} + \cdots + 2b_1 + b_0, \text{ where } b_q = 1, b_i \in \{0, 1\}.$$

Let us denote the decreased-by-1 length of the binary notation of n as $\lambda(n) = q$, and the sum of figures, i.e. the amount of those b_i numbers that are equal to 1 as $\nu(n)$. It is clear that $\lambda(n) = q = [\log_2 n]$. The desired number of steps in a minimal algorithm we denote as $l(n)$.

As we have already noted, Problem 4 turned out to be "a hard nut to crack" in the general case, and vast research field was formed around it. We will give here (without proof) a number of theorems, which demonstrate the development and improvement of fast algorithms. The theorems and their proofs, as well as some exercises, can be found in [6].

The following theorem is based on the direct investigation of the binary technique properties.

Theorem 1
The inequalities $\lambda(n) \le l(n) \le \lambda(n) + \nu(n) - 1$ are met.

Then we can easily obtain the answer to Problem 2 for any n, specifically, with due consideration for the "shift":

$$t(n) = \lambda(n) + \nu(n) = [\log_2 n] + \nu(n).$$

It turns out that the binary technique is optimal when $\nu(n) \le 3$.

Theorem 2

$$l(2^p + 2^q + 2^r) = p + 2, \text{ for } p > q > r.$$

With $\nu(n) = 4$, all cases are also described by Theorem 1 and by the following theorem.

Theorem 3 (D. Knuth)
Let $\nu(n) \ge 4$, then

1. $l(2^p + 2^q + 2^r + 2^t) = p + 2$, where $p > q > r > t$, and one of the four cases is met:
   a) $p - q = r - t$ (Example: $n = 15$);
   b) $p - q = r - t + 1$ (Example: $n = 23$);
   c) $p - q = 3$, $r - t = 1$ (Example: $n = 39$);
   d) $p - q = 5$, $q - r = r - t = 1$ (Example: $n = 135$).
2. Otherwise, $\lambda(n) + 3 \le l(n)$.

To prove the following theorem, the m-ary method should already be applied, because, as we have briefly mentioned before, this algorithm is a "modernization" of the binary technique.

Theorem 4
For $k < \log_2 \log_2 n$ the inequality

$$l(n) \leq (1 + 1/k)(\lambda(n) + 1) + 2^{k-1} - k + 2$$

is met.

Corollary 1

$$\lim_{n \to \infty} \frac{l(n)}{\log_2 n} = 1.$$

Examining the method of multipliers, we have

Theorem 5 (E. de Jonquieres)
The inequality $l(nm) \leq l(n) + l(m)$ is held.

If a particular case of the previous theorem is taken for $m = 2$, it can be easily seen that $l(2n) \leq l(n) + 1$. And if a deeper look is taken at the figures, it turns out that here actually the equality is held, since intuitive comprehension hints that nothing can be faster than the doubling. Nevertheless, it is not as it seems! There are such numbers that $l(2n) = l(n)$! Moreover, incredible as it may seem, but recently (see [7]) it was computed that for $n = 375494703$, $34 = l(2n) < l(n) = 35$ is held! You can attempt to check the existence of such n and m numbers that $l(nm) < l(n)$ by solving the exercise.

Exercise 1
Check that for $n = \frac{2^{13}+1}{3} = 2731$ and $m = 3$ numbers the $l(nm) < l(n)$ is held.

Exercise 2
Prove $l(n) \geq \log_2 n$ inequality and give examples of its transformation into equality.

Exercise 3
Give examples of the cases when the method of multipliers is better than

the binary technique, and opposite examples. Show that such examples are IO.

The possibility of combining the binary technique and the method of multipliers is applied in the proof of the following theorem:

Theorem 6 (W. Hansen)

$$l(2^q + km) \leq q + \nu(k) + \nu(m) - 1, \text{ if } \lambda(k) + \lambda(m) \leq q.$$

We will mention one more technique. Visually, this technique is different from the binary technique and the method of multipliers, and is related to the plotting of graph trees. Nevertheless, these constructions are easily programmed and we can say that these trees bear good fruit.

Let us construct a tree based on the binary algorithm, see Fig. 1.

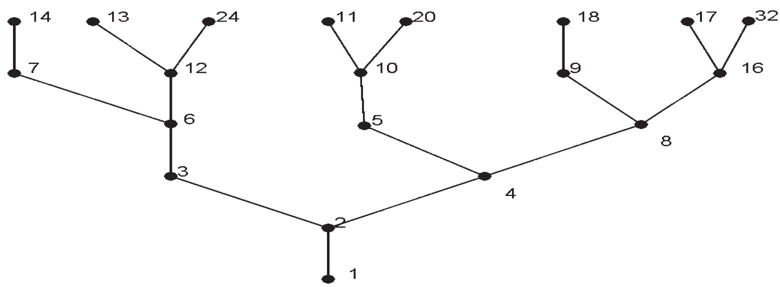

Fig. 1

Let us plot another tree with slightly differing shape, Fig. 2.

The tree development algorithm is sufficiently simple:

- the first level starts from 1;
- if the tree kth level is constructed, the $k+1$th level is constructed as follows: we take alternately, starting from the *highest-order node*, each m node at the kth level and add on only those $2m = m + a_{k-1}, \ldots, m + a_2, m + a_1, m + 1$ nodes which are yet absent from the tree, and so that $m = a_{k-1}, \ldots, a_2, a_1, 1$ denotes the route from the tree root to our node.

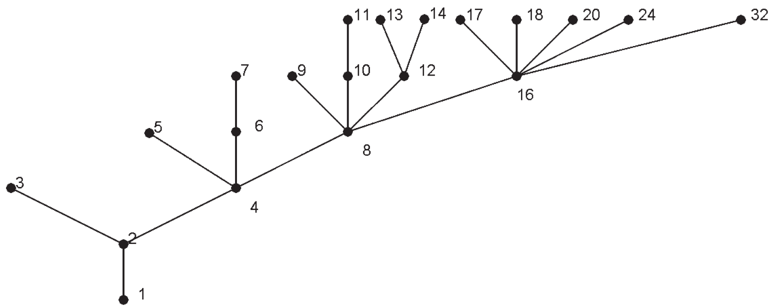

Fig. 2

The tree, evidently, provides the alternative technique of raising to power. Note that in this algorithm, the chain length coincides with the chain length calculated by the binary technique. Try to prove it.

It turns out that minor variations in the tree plotting result in the technique that operates over a large number array faster than the binary technique and the method of multipliers. So, for instance, the initial value where the method of multipliers is faster than the power tree method, is $n = 19879$. It stands to reason that such calculations can be performed exclusively by means of computers, because all said algorithms can be programmed.

Let us copy the tree plotting algorithm for the *power tree method*. We will view the tree as genuine mathematicians, i.e. we turn its roots upward. Thus:

- the first level starts from 1;
- if the tree kth level is constructed, the $k+1$th level is constructed as follows: we take alternately, starting from the *lowest-order node*, each m node at the kth level and add on only those $m+1, m+a_1, m+a_2, \ldots, m+a_{k-1} = 2m$ nodes which are yet absent from the tree, and so that $m = a_{k-1}, ..., a_2, a_1, 1$ denotes the route from the tree root to our node.

Fig. 3 shows the power tree constructed up to the 7th level.

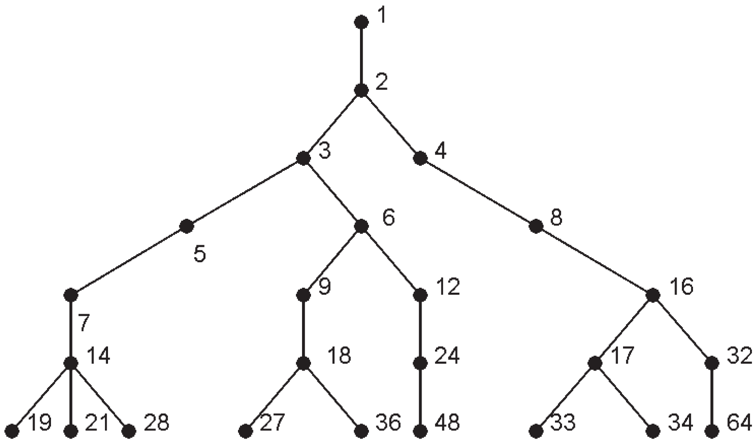

Fig. 3

The availability of such diversified algorithms and the application of combined techniques has not yet provided the ultimate solution for this seemingly simple problem. And even in a particular case there is the unsolved *Scholz-Brauer conjecture* that states that

$$l(2^n - 1) \leq n - 1 + l(n).$$

Since $2^n - 1$ has minimal number of unities in its binary presentation, this particular case is interesting because it is the worst case for the binary technique. Computations have shown that for $1 \leq n \leq 64$, the equality (see [7]) holds.

The above discussed algorithms can be represented as pictorial terms of addition chains. But we will not deal with it within the framework of our narration. An inquisitive reader can find the data related to the addition chains in the below cited literature (see [6], [7], [8]).

In our discussion, we have not actually touched upon Problem 3. The point is that in most cases, by constructing the optimal algorithm of power finding by means of multiplication and division, such an algorithm can be constructed that will cope with the problem, using only multiplication and the same number of steps. In the forgoing, we have constructed the division-free example for obtaining x^{15} in 5 steps. We offer our readers to plot the 12-step algorithm for obtaining x^{1000}, using multiplication and division, and the 12-step algorithm for obtaining x^{1000}, using multiplication only. The estimate given in clause b) of this Problem is much weaker than that formulated in Theorem 4 and in Corollary 1. For the sake of truth it should be mentioned that there are examples when raising to a power, using multiplication and division, takes fewer steps than using the multiplications only. For example, it is $2^n - 1$ that was already mentioned in the Scholz-Brauer conjecture. Using multiplication and division, the same plotting can be made in no more than $n + 1$ steps.

Exercise 4
Give some additional examples, when raising to a power using multiplication and division takes fewer steps than using the multiplications only.

Let us digress from the power topic and take a look at Problem 5. Again, start from the end, i.e. from clause b). At this point, our reader should have exclaimed "the binary technique"! We invite our readers to handle the clause solution themselves, or to be more precise, to find a complete analogy with Problem 1, dealing with a plummet and sugar.

Clause a) prepares a surprise for us. Of course, we can apply the binary technique and drop a glass marble from the 50th storey. And if it does not break down, then under the unfavorable outcome, we will be forced to conduct another 49 tests. We cannot jeopardize the remaining marble and further apply the binary technique; therefore, we will be forced to gradually drop the remaining marble from the 1st, 2nd, etc. floor, probably up to the 49th floor. In other words, under the most unfavorable outcome, we are forced to make 50 attempts. Naturally, the suspicion of the algorithm optimality definitely arises.

A decimal notation lover would offer to drop the first marble from the 10th, 20th, etc., 90th, and 100th floor, and check the remaining 9-floor interval using the second marble. Such an algorithm, even under the most unfavorable outcome, helps solve the problem in 19 attempts. And though the resulting algorithm is not optimal, it became clear what should be done. At this point we will give the reader the opportunity of finding the problem solution, though making a small remark. The answer found for the 100-storey building, will also fit the 101–105-storey buildings.

Varying the number of storeys, marbles, and even the number of attempts, we have an interesting problem, serving as generalized Problem 5 to be investigated by the students.

Problem 7
We have $B = const$ for similar glass marbles and a skyscraper. You can drop a marble from various storeys to clarify at what storey your marble breaks down from falling (for instance, it gets broken by falling from the 5th floor, and still remains intact by falling from the 4th floor). To save the means, you have preset the constant number $T = const$ of such tests ($T \geq B$). Demonstrate the optimal test mechanism and find out the maximal skyscraper storey that can be checked for sure.

It is not entirely clear how Problem 6 is related to this story. Before we arrive at our techniques, we should make two significant observations. The first one is that we can get rid of a circumference, and the second one is that we must construct a certain iteration process.

Let us represent the trees by points. Having learned the tree age, we will mark the corresponding point by a number. Without losing the generality, we can assume that all points (trees) are at equal distance from each other. At the two initial steps we will learn the age of two arbitrary trees and mark the corresponding circumference points by figures. Then we can perform the well-known operation of circumference cutting at the point with smaller value and the circumference straightening into a segment. We obtain a segment with points placed at equal distances (we will consider this distance as 1). The segment ends are marked by similar numbers, and one of the points inside the segment is marked by the number larger than the numbers at the segment ends. The circumference has vanished! And we can make our constructions on the integer straight line.

We can attempt to construct the iteration process. Let us assume that at a certain step we have a segment with the following properties:

- the segment ends are marked by a, b numbers;
- inside the segment we have the marked point c;

– $a < c$, $b < c$ is held.

Note that after two initial steps when we have cut the circumference, we obtained a segment with the required properties. If we succeed to find (construct) the segment with similar properties but of lesser length at the following step, it will become clear that at the finite number of such steps the process will end in the construction of the segment, having length 2, and in finding three points (trees) that satisfy the problem conditions.

Let us make the following step. So, if the $[a, b]$ segment has larger than 2 length, we take one of the segments of $[a, c]$ or $[c, b]$ and mark a certain point d; without losing generality, we can assume that $d \in [a, c]$. If $d < c$, then the $[d, b]$ segment with marked point c is of lesser length and has all required properties. If $d > c$, then the $[a, c]$ segment with marked point d possesses all required properties.

If a reader could make sense of the given constructions, he can again exclaim the magic words of "binary algorithm". As long as we did not impose any condition on point d except for being inside the $d \in [a, c]$ segment, then why cannot we choose it to be exactly at the dead center of a segment? But since we are dealing with an integer straight line, we will not always succeed to select the point at the dead center. Then, the reader tells us, it should be as close to the middle of the segment as possible. Thus, we obtain the algorithm that will assist us in sufficiently fast finding of three required points.

The reader can also say that we did see that the binary algorithm is not necessarily the fastest one. Actually, what if we do not bisect the segment but, instead, adhere to a certain proportion? We can obtain various algorithms in this way, and which of them is the fastest?

Let us try to estimate the algorithm operation rate under the most unfavorable conditions. Whence are the unfavorable conditions in this case? The point is that in these constructions, the lengths of $[a, c]$ and $[d, b]$ segments may differ as a function of d value measurements, and thus, the rate of achieving our objective may also vary. It is evident that here we have a certain similarity to Problem 5.

Let us examine several operation steps of the binary algorithm and the algorithm with k proportion factor, see Fig. 4. Let the initial segment length be l. To construct a proportional algorithm, we take $k > 1 - k$ so that $kl \leq k(1-k)l + (1-k)l$. Such constructions are possible if $\frac{1}{2} < k \leq \frac{1}{2}(\sqrt{5}-1)$.

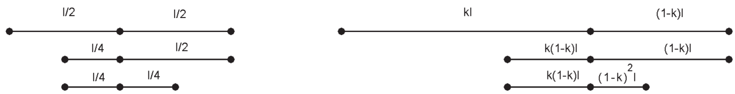

Fig. 4

Fig. 4

After the first step the binary algorithm appears to operate faster, but let us observe what happens after two steps. After two steps the binary algorithm gives a segment, having $l/2$ length. The proportional algorithm gives, after two steps, the segment of $(1-k)l$ length. Note that $\left(1 - \frac{1}{2}(\sqrt{5}-1)\right) l < \frac{l}{2}$. It means that with k values close to $\frac{1}{2}(\sqrt{5}-1)$ the proportional algorithm will operate faster than the binary one. And the golden-section proportional algorithm would have been the fastest. The best rational approximation to the golden section is known to be achieved by means of such numerator – denominator fractions that are the sequential Fibonacci numbers. If we introduce a correction for integer-number points in the reported reasoning, then the Fibonacci-number-based algorithm looks quite natural.

We denote the minimal amount of steps (amount of trees) necessary to find three such trees that certainly satisfy the condition (even in the most unfavorable situation) as $h(n)$.

Exercise 5
Check that for $n = 8$ the binary algorithm brings the desired aim in 6 steps (the age of 6 trees should be examined). Find the algorithm that will bring the desired aim in 5 steps. Prove that under unfavorable conditions it is insufficient to know the age of 4 trees.

Exercise 6
The age of how many trees (how many steps are necessary) should be determined from the binary algorithm for $n = 2^m$?

Exercise 7
Construct the algorithm for $n = F_m$, where F_m is the mth member in the Fibonacci sequence.

Prove that $h(F_m) \leq m - 1$.

Exercise 8
Based on the constructions of Exercise 7, plot the algorithm for each $F_{m-1} < n \leq F_m$, where F_m is the mth member of the Fibonacci sequence.

Prove that $h(n) \leq m - 1$.

Proof: The proof is made by induction on m. The induction base with $n = 3$, $m = 4$ is evidently met.

Suppose that for all $n = k$ such that $k \leq F_{m-1}$, $h(k) \leq m-2$ is met. Let us inspect $F_{m-1} < n \leq F_m$. Following our algorithm, we conduct the first test with any tree, then we check the tree found at the F_{m-1} distance from this tree. Now we perform the operation of circumference cutting at the point with lesser value, and the circumference straightening into a segment, see Fig. 5.

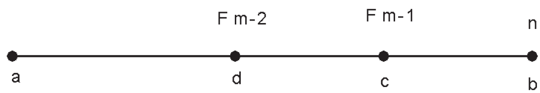

Fig. 5

Fig. 5

We have $a < c$, $b < c$. Then we test the tree numbered F_{m-2}.

If $d < c$, then the $[d, b]$ segment with marked c point has the $n - F_{m-2}$ length. Since $n - F_{m-2} \leq F_m - F_{m-2} = F_{m-1}$, then the induction proposition is applicable. Thus, in this case $h(n) \leq m - 2 + 1 \leq m - 1$.

If $d > c$, then the $[a, c]$ segment with marked d point has all the required properties. This segment has the F_{m-1} length, and the induction proposition is also applicable here. Therefore, in this case $h(n) \leq m - 2 + 1 \leq m - 1$ is also true.

Quod erat demonstrandum.

4 Problems to be researched

We have considered the algorithms related to numerical sequences. However, in modern computers, graphic editors are also installed. For this reason we offer you some new problems.

Problem 8 Graphic editor problem (for study)
Graphic editor can perform the following 2D copying operations:

1. Copy and at the same time reproduce a figure with regard to a selected axis;
2. Copy a current figure or a figure from memory and displace it concurrently.

During such operations the old figure remains in its place.

2D figures obtained can be saved in computer memory.

What minimal number of copying operations is necessary to obtain the target figure from the reference one, if:

1. The reference figure is a 1×1 square, and the target figure is a $n \times n$ square;
2. The reference figure is a 1×1 square, and the target figure is a $n \times m$ rectangle;
3. The reference figure is an equilateral triangle with side 1, and the target figure is a triangle with n side length.

For instance the 2×2 square can be produced from the 1×1 square, using these two operations:

First, aided by symmetry or by parallel transfer we produce a 1×2 domino die, then, in a similar way, either by parallel transfer or by symmetry relative to the long side, we produce the 4×4 square.

It is clear that when the reference figure is a square we can apply the results obtained for Problem 4 in this problem.

Exercise 9
Produce the estimates similar to those produced in Theorem 1 and Theorem 5 for the reference square figure.

In the following problem we propose to replace doubling by multiplication by a set number p. Which algorithm will be the fastest: the p-ary technique, the method of multipliers, or the method of the power tree? The authors cannot predict the potential outcome.

Problem 9 (for study)

a) Peter and Sam have calculators that perform two operations. Peter's calculator can:
   - Multiply a number by natural number p;
   - Increase a number by 1.

   Sam's calculator can:
   - Multiply a number by natural number q;
   - Increase a number by 1.

   $1 < p < q$. At the starting moment both calculators show zeros. Find all natural numbers that Peter can count in fewer operations than Sam.

b) Solve the problem for $p = 2$, $q = 3$.

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Tournament of the Towns (Selections from the A-Level, Fall 2010)

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1. There are 100 points on the plane. All 4950 pairwise distances between two points have been recorded.

(a) A single record has been erased. Is it always possible to restore it using the remaining records?

(b) Suppose no three points are on a line, and k records were erased. What is the maximum value of k such that restoration of all the erased records is always possible?

2. Two dueling wizards are at an altitude of 100 above the sea. They cast spells in turn, and each spell is of the form decrease the altitude by a for me and by b for my rival" where a and b are real numbers such that $0 < a < b$. Different spells have different values for a and b. The set of spells is the same for both wizards, the spells may be cast in any order, and the same spell may be cast many times. A wizard wins if after some spell, he is still above water but his rival is not. Does there exist a set of spells such that the second wizard has a guaranteed win, if the number of spells is

(a) finite;

(b) infinite?

3. The quadrilateral $ABCD$ is inscribed in a circle with center O. The diagonals AC and BD do not pass through O. If the circumcentre of triangle AOC lies on the line BD, prove that the circumcentre of triangle BOD lies on the line AC.

4. In acute triangle ABC, an arbitrary point P is chosen on altitude AH. Points E and F are the midpoints of sides CA and AB respectively. The perpendiculars from E to CP and from F to BP meet at point K. Prove that $KB = KC$.

5. Merlin summons the n knights of Camelot for a conference. Each day, he assigns them to the n seats at the Round Table. From the second day on, any two neighbours may interchange their seats if they were not neighbours on the first day. The knights try to sit in some cyclic order which has already occurred before on an earlier day. If they succeed, then the conference comes to an end when the day is over. What is the maximum number of days for which Merlin can guarantee that the conference will last?

6. Each cell of a 1000×1000 table contains 0 or 1. Prove that one can either cut out 990 rows so that at least one 1 remains in each column, or cut out 990 columns so that at least one 0 remains in each row.

Solutions

1. Solution by Central Jury.

(a) This is not always possible. Suppose the record of the distance AB is lost. If the other 98 points all lie on a line ℓ, we cannot tell whether A and B are on the same side of ℓ or on opposite sides of ℓ. Thus the lost record cannot be restored from the remaining ones.

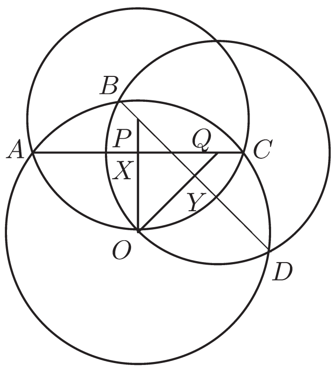
(b) The answer is 96. Suppose 97 records are erased. All of them may be associated with a point A so that we only know the distances AB and AC, where B and C are 2 of the other 99 points. A does not lie on BC as no three of the 100 points lie on a line. Now we cannot determine whether A is on one side or the other side of the line BC. Suppose at most 96 records are erased. Construct a graph with 100 vertices representing the 100 points. Two vertices are joined by an edge if the record of the distance between the two points they represent is erased. The graph has at most 96 edges, and therefore at least 4 components. Take four vertices A, B, C and D, one from each component. The pairwise distances between the points A, B, C and D are on record, so that their relative position can be determined. For any other vertex P, it is in the same component with only one of these four vertices. Hence the distance between the point P and three of the points A, B, C and D are on record. This is enough to determine the position of the point relative to the points A, B, C and D. It follows that all erased records may be restored.

2. (a) The answer is no. With a finite number of spells, there is one for which $b - a$ is maximum. If the first wizard keeps casting this spell, the best that the second wizard can do is to maintain status quo by casting the same spell. Hence the second wizard will hit the water first, giving the first wizard a win.

(b) The answer is yes. In the n-th spell, let $a = \frac{1}{n}$ and $b = 100-\frac{1}{n}$. By symmetry, we may assume that the first wizard casts the n-th spell. He is then $100 - \frac{1}{n}$ above water while the second wizard is $\frac{1}{n}$ above water. However, the second wizard wins immediately by casting the $(n + 1)$-st spell. He will still be $\frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)}$ above water while the first wizard is submerged in water since $(100 - \frac{1}{n}) - (100 - \frac{1}{n+1}) = -\frac{1}{n(n+1)}$.

3. Let P be the circumcentre of triangle OAC. Then PO is perpendicular to AC, intersecting AC at X. Let the line through O perpendicular to BD intersect BD at Y and AC at Q. We claim that Q is the circumcentre of triangle OBD. By Pythagoras' Theorem,

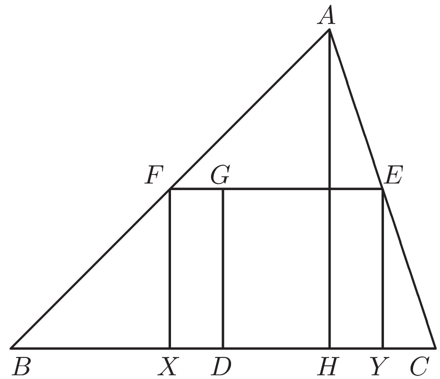
$$\begin{aligned}
QD^2 &= QY^2 + DY^2 \\
&= QY^2 + (OD^2 - OY^2) \\
&= QY^2 + OA^2 - (OP^2 - PY^2) \\
&= OA^2 - AP^2 + PQ^2 \\
&= QX^2 + OA^2 - (AP^2 - PX^2) \\
&= QX^2 + (OA^2 - AX^2) \\
&= QX^2 + OX^2 \\
&= QO^2.
\end{aligned}$$



4. First Solution:

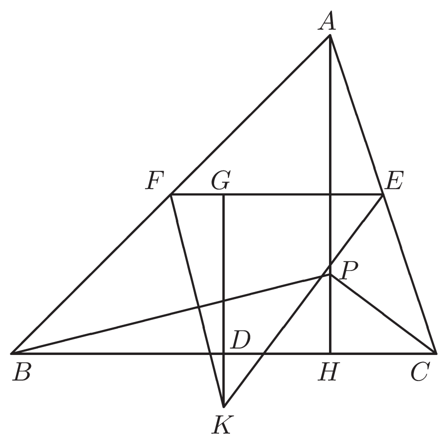
We first prove an auxiliary result. Let the line through the midpoint D of BC and perpendicular to BC cut EF at G. Then $BH \cdot FG = CH \cdot EG$. Drop perpendiculars FX and EY from F and E to BC respectively. Then X is the midpoint of BH and Y is the midpoint of CD. Since triangles FXD and EYC are congruent, we have $XD = YC = YH$ so that $XH = YD$. Now $BH \cdot FG = 2XH \cdot XD = 2YH \cdot YD = CH \cdot EG$.

Returning to the main problem, let the line through F perpen-



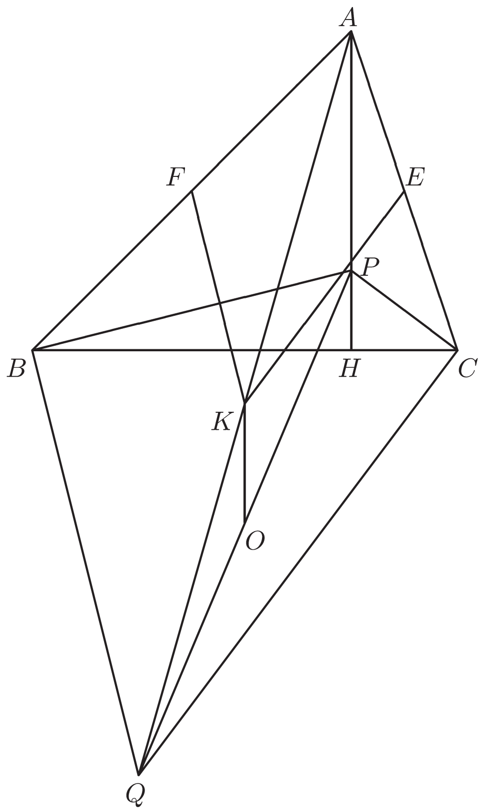
dicular to BP intersect the line DG at K_1. Triangles FGK_1 and PHB are similar. Hence $GK_1 = \frac{FG \cdot BH}{PH}$. Let the line through E perpendicular to CP intersect the line DG at K_2. We can prove in an analogous manner that $GK_2 = \frac{EG \cdot CH}{PH}$. By the auxiliary result, $GK_1 = GK_2$. Hence K_1 and K_2 is the same point K. Since K lies on the perpendicular bisector of BC, we have $KB = KC$.

Second Solution by Chun-Yu Yang.



Extend AK to Q so that K is the midpoint of AQ. Since F is the midpoint of AB, BQ is parallel to FK, which is perpendicular to BP. Hence $\angle PBQ = 90°$. Similarly, $\angle PCQ = 90°$. Hence the midpoint O of PQ is the circumcentre of the cyclic quadrilateral $BPCQ$. Now OK is parallel to PA, which is perpendicular to BC. Hence K lies on the perpendicular bisector of BC, so that $KB = KC$.

5. Solution by Central Jury.
We may assume that the Knights are seated from 1 to n in cyclic clockwise order on day 1. Then seat exchanges are not permitted



between Knights with consecutive numbers (1 and n are considered consecutive). We construct an invariant for a cyclic order called the winding number as follows. Merlin has n hats numbered from 1 to n from top to bottom. He starts by giving hat 1 to Knight 1. Then he continues in the clockwise order round the table until he gets to Knight 2, when he will give him hat 2. After he has handed out all the hats, Merlin returns to Knight 1 which is his starting point. The number of times he has gone round the table is called the winding number of the cyclic order. For instance, the winding number of the cyclic order 1 4 7 2 3 6 5 is 4: (1,2,3)(4,5)(6)(7). Suppose two adjacent Knights change seats. If neither is Knight 1, the hats handed out in each round remain the same, so that the winding number remains constant. Suppose the seat exchange is between Knight 1 and Knight h, where $h \neq 2$ or n. Then Knight h either becomes the first Knight to get a hat in the next cycle instead of the last Knight to get a hat in the preceding cycle, or vice versa. Still, the winding number remains constant. On the k-th day of the conference, $1 \leq k \leq n-1$, Merlin can start by having the Knights sit in the cyclic order $k, k-1, \ldots, 2, 1, k+1, k+2, \ldots, n$. It is easy to verify that the winding number of the starting cyclic order on the k-th day is k. It follows that no cyclic order can repeat on two different days within the first $n-1$ days. Therefore, Merlin can make the conference last at least n days.

We claim that given any cyclic order, it can be transformed into one of those with which Merlin starts a day. Then the Knights can make the conference end when the n-th day is over. Suppose the cyclic order is not one with which Merlin starts a day. We will push Knight 2 forward in clockwise direction until he is adjacent to Knight 1. This can be accomplished by a sequence of exchanges if he does not encounter Knight 3 along the way. If he does, we will push both of them forward towards Knight 1. Eventually, we will have Knights $2, 3, \ldots, h$ and 1 in a block. Now $h < n$ as otherwise the initial cyclic order is indeed one of those with which Merlin starts a day. Hence we can push Knight 1 counter-clockwise so that he is adjacent to Knight 2. We now attempt to put Knight 3 on the other side of Knight 2. As before, we have Knights $3, 4, \ldots, \ell, 2, 1$ in a block. If $\ell < n$, we can push Knights 2 and 1 counter-clockwise towards Knight 3. If $\ell = n$, Knight 1 cannot get past, but we notice that we have arrived at one of the cyclic orders with which Merlin starts a day. This justifies the claim.

6. **Solution by Brian Chen.**
Let $S(p,q)$ denote the following statement.
"In any binary $a \times b$ table with $ab \leq p$, one of the following is true.
(A) There exists an $a \times q$ subtable with at least one 0 in each row.
(B) There exists a $q \times b$ subtable with at least one 1 in each column.
The q rows or columns may be chosen arbitrarily."
We wish to prove that $S(1000000, 10)$ is true. We first examine $S(4,1)$. The relevant tables are 1×4, 1×3, 1×2, 1×1, 2×1, 3×1, 4×1 and 2×2. In the first four, if there is at least one 0, then (A) is true. Otherwise, (B) is true. In the next three, if there is at least one 1, then (B) is true. Otherwise, (A) is true. In the last one, if there are no 0s in the first row, then (B) is true. Suppose there is at least one 0 in the first row. If there is at least one 0 in the second row as well, then (A) is true. Otherwise, (B) is true. It follows that $S(4,1)$ is true. We claim that $S(p,q)$ implies $S(4p, q+1)$. Consider any $a \times b$ table with $ab \leq 4p$. Let x be the minimum number of 1s in any row and y be the minimum number of 0s in any column. Then the total number of 1s is at least ax and the total number of 0s is at least by. It follows that $ax + by \leq ab$. By the Arithmetic-Geometric Means Inequality, $\sqrt{(ax)(by)} \leq \frac{ax+by}{2} \leq \frac{ab}{2}$. Hence $(ax)(by) \leq \frac{(ab)^2}{4}$ so that $xy \leq p$. Let R be a row with exactly x 1s and C be a column with exactly y 0s. Consider the $y \times x$ table whose rows have 0s at the intersections with C and whose columns have 1s at the intersections with R. We are assuming that $S(p,q)$ is true, so that either (A) or (B) holds for this table. If (A) holds, adding C would make (A) hold for the $a \times b$ table. If (B) holds, adding R would make (B) hold for the $a \times b$ table. This justifies the claim. From $P(4,1)$, we can deduce in turns $S(16,2)$, $S(64,3)$ and so on, up to $S(1048576, 10)$. This clearly implies $S(1000000, 10)$.

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The 4th Middle European Mathematical Olympiad 9–15 September 2010 Strečno, Slovakia

The 4th Middle European Mathematical Olympiad (MEMO) was held on 9–15 September in Strečno, Slovakia. Main competition organisers in 2010 have been: Union of Slovak Mathematicians and Physicists, Central Committee of Slovak MO, University of Žilina and society Iuventa with a financial support from the Ministry of Education, Science, Research and Sport of the Slovak Republic. At this event 10 countries from the region of Middle-Europe (Austria, Croatia, the Czech Republic, Germany, Hungary, Lithuania, Poland, Slovakia, Slovenia and Switzerland) took part. Each team consisted as usual of up to six members (59 contestants altogether).

In general, the MEMO competition consists of two exam papers which are sat over consecutive days. On the first day competitors individually solve four problems from the areas of algebra, combinatorics, geometry and number theory with the time limit of five hours to solve them. For the second day organizers have prepared the set of 8 problems for team competition, with the time limit of five hours again.

Most of the teams arrived in Poznań on 9 September. The next day, the *Jury* consisting of the team leaders terminated their preparation of the MEMO paper. The Problem Selection Committee composed a shortlist of problems sent from all participating countries. On their final meeting the day before the competition started, the jury selected the final 12 problems for the competition and translated them to the mother language of the contestants.

While the leaders were preparing the final version of the problems, the contestants along with the deputy leaders visited the Streno castle and looked up to great outdoors of Western Slovakia from rafts on the river Váh. In the following days organizers prepared trips to the national park Malá Fatra.

On Saturday and Sunday, 11–12 September, contestants sat the papers. Each day after the exam, leaders and deputy leaders assessed the work of the students from their own countries. Final evaluation was done after coordination with the local coordinators.

Finally we present the complete set of problems. All results and solutions can be found on the website of the competition

http://memo2010.skmo.cz/.

Individual Competition

1. Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$, we have

$$f(x+y) + f(x)f(y) = f(xy) + (y+1)f(x) + (x+1)f(y).$$

(*Czech Republic*)

2. All positive divisors of a positive integer N are written on a blackboard. Two players A and B play the following game taking alternate moves. In the first move, the player A erases N. If the last erased number is d, then the next player erases either a divisor of d or a multiple of d. The player who cannot make a move loses. Determine all numbers N for which A can win independently of the moves of B.

(*Poland*)

3. We are given a cyclic quadrilateral $ABCD$ with a point E on the diagonal AC such that $AD = AE$ and $CB = CE$. Let M be the center of the circumcircle k of the triangle BDE. The circle k intersects the line AC in the points E and F. Prove that the lines FM, AD, and BC meet at one point.

(*Switzerland*)

4. Find all positive integers n which satisfy the following two conditions:

(i) n has at least four different positive divisors;

(ii) for any divisors a and b of n satisfying $1 < a < b < n$, the number $b - a$ divides n.

(*Slovenia*)

Team Competition

1. Three strictly increasing sequences

$$a_1, a_2, a_3, \ldots, \qquad b_1, b_2, b_3, \ldots, \qquad c_1, c_2, c_3, \ldots$$

of positive integers are given. Every positive integer belongs to exactly one of the three sequences. For every positive integer n, the following conditions hold:

(i) $c_{a_n} = b_n + 1$;

(ii) $a_{n+1} > b_n$;

(iii) the number $c_{n+1}c_n - (n+1)c_{n+1} - nc_n$ is even.

Find a_{2010}, b_{2010}, and c_{2010}.

(*Lithuania*)

2. For each integer $n \geq 2$, determine the largest real constant C_n such that for all positive real numbers $a_1, \ldots, a_n$, we have

$$\frac{a_1^2 + \cdots + a_n^2}{n} \geq \left(\frac{a_1 + \cdots + a_n}{n}\right)^2 + C_n \cdot (a_1 - a_n)^2.$$

(*Switzerland*)

3. In each vertex of a regular n-gon there is a fortress. At the same moment each fortress shoots at one of the two nearest fortresses and hits it. The *result of the shooting* is the set of the hit fortresses; we do not distinguish whether a fortress was hit once or twice. Let $P(n)$ be the number of possible results of the shooting. Prove that for every positive integer $k \geq 3$, $P(k)$ and $P(k+1)$ are relatively prime.

(*Czech Republic*)

4. Let n be a positive integer. A square $ABCD$ is partitioned into n^2 unit squares. Each of them is divided into two triangles by the diagonal parallel to BD. Some of the vertices of the unit squares are colored red in such a way that each of these $2n^2$ triangles contains at least one red vertex. Find the least number of red vertices.

(*Slovenia*)

5. The incircle of the triangle ABC touches the sides BC, CA, and AB in the points D, E, and F, respectively. Let K be the point symmetric to D with respect to the incenter. The lines DE and FK intersect at S. Prove that AS is parallel to BC.

(*Poland*)

6. Let A, B, C, D, E be points such that $ABCD$ is a cyclic quadrilateral and $ABDE$ is a parallelogram. The diagonals AC and BD intersect at S and the rays AB and DC intersect at F. Prove that $\angle AFS = \angle ECD$.

(*Croatia*)

7. For a nonnegative integer n, define a_n to be the positive integer with decimal representation

$$1\underbrace{0\ldots0}_{n}2\underbrace{0\ldots0}_{n}2\underbrace{0\ldots0}_{n}1.$$

Prove that $a_n/3$ is always the sum of two perfect cubes but never the sum of two perfect squares.

(*Switzerland*)

8. We are given a positive integer n which is not a power of 2. Show that there exists a positive integer m with the following two properties:

(i) m is the product of two consecutive positive integers;

(ii) the decimal representation of m consists of two identical blocks of n digits.

(*Poland*)

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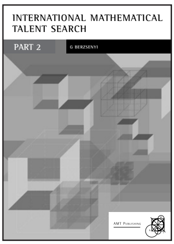

International Mathematical Talent Search Part 2
G Berzsenyi

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He seeks to encourage problem solving as an intellectual habit and this book contains many interesting and some unusual problems, many with detailed backgrounds and insights. This collection contains the problems and solutions of rounds 21-44 of the IMTS.

Both Part 1 & Part 2 are aimed at advanced, senior students at Year 10 level and above.

Bundles of Past AMC Papers

Past Australian Mathematics Competition papers are packaged into bundles of ten identical papers in each of the Junior, Intermediate and Senior divisions of the Competition. Schools find these sets extremely valuable in setting their students miscellaneous exercises.

AMC Solutions and Statistics
Edited by PJ Taylor

This book provides, each year, a record of the AMC questions and solutions, and details of medallists and prize winners. It also provides a unique source of information for teachers and students alike, with items such as levels of Australian response rates and analyses including discriminatory powers and difficulty factors.

Australian Mathematics Competition Book 1 1978–1984
Edited by W Atkins, J Edwards, D King, PJ O'Halloran & PJ Taylor

This 258-page book consists of over 500 questions, solutions and statistics from the AMC papers of 1978-84. The questions are grouped by topic and ranked in order of difficulty. The book is a powerful tool for motivating and challenging students of all levels. A must for every mathematics teacher and every school library.

Australian Mathematics Competition Book 2 1985–1991
Edited by PJ O'Halloran, G Pollard & PJ Taylor

Over 250 pages of challenging questions and solutions from the Australian Mathematics Competition papers from 1985-1991.

Australian Mathematics Competition Book 3 1992–1998
W Atkins, JE Munro & PJ Taylor

More challenging questions and solutions from the Australian Mathematics Competition papers from 1992-1998.

Australian Mathematics Competition Book 3 CD
Programmed by E Storozhev

This CD contains the same problems and solutions as in the corresponding book. The problems can be accessed in topics as in the book and in this mode is ideal to help students practice particular skills. In another mode students can simulate writing one of the actual papers and determine the score that they would have gained. The CD runs on Windows platform only.

Australian Mathematics Competition Book 4 1999–2005
W Atkins & PJ Taylor

More challenging questions and solutions from the Australian Mathematics Competition papers from 1999-2005.

Australian Mathematics Competition Primary Problems & Solutions Book 1 2004–2008
W Atkins & PJ Taylor
This book consists of questions and full solutions from past AMC papers and is designed for use with students in Middle and Upper Primary. The questions are arranged in papers of 10 and are presented ready to be photocopied for classroom use.

Problem Solving via the AMC
Edited by Warren Atkins

This 210-page book consists of a development of techniques for solving approximately 150 problems that have been set in the Australian Mathematics Competition. These problems have been selected from topics such as Geometry, Motion, Diophantine Equations and Counting Techniques.

Methods of Problem Solving, Book 1
Edited by JB Tabov & PJ Taylor

This book introduces the student aspiring to Olympiad competition to particular mathematical problem solving techniques. The book contains formal treatments of methods which may be familiar or introduce the student to new, sometimes powerful techniques.

Methods of Problem Solving, Book 2
JB Tabov & PJ Taylor

After the success of Book 1, the authors have written Book 2 with the same format but five new topics. These are the Pigeon-Hole Principle, Discrete Optimisation, Homothety, the AM-GM Inequality and the Extremal Element Principle.

Mathematical Toolchest
Edited by AW Plank & N Williams

This 120-page book is intended for talented or interested secondary school students, who are keen to develop their mathematical knowledge and to acquire new skills. Most of the topics are enrichment material outside the normal school syllabus, and are accessible to enthusiastic year 10 students.

International Mathematics — Tournament of Towns (1980-1984)
Edited by PJ Taylor

The International Mathematics Tournament of the Towns is a problem-solving competition in which teams from different cities are handicapped according to the population of the city. Ranking only behind the International Mathematical Olympiad, this competition had its origins in Eastern Europe (as did the Olympiad) but is now open to cities throughout the world. This 115-page book contains problems and solutions from past papers for 1980-1984.

International Mathematics — Tournament of Towns (1984-1989)
Edited by PJ Taylor

More challenging questions and solutions from the International Mathematics Tournament of the Towns competitions. This 180-page book contains problems and solutions from 1984-1989.

International Mathematics — Tournament of Towns (1989-1993)
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This 200-page book contains problems and solutions from the 1989-1993 Tournaments.

International Mathematics — Tournament of Towns (1993-1997)
Edited by PJ Taylor

This 180-page book contains problems and solutions from the 1993-1997 Tournaments.

International Mathematics — Tournament of Towns (1997-2002)
Edited by AM Storozhev

This 214-page book contains problems and solutions from the 1997-2002 Tournaments.

Challenge! 1991 – 1998
Edited by JB Henry, J Dowsey, AR Edwards, L Mottershead, A Nakos, G Vardaro & PJ Taylor

This book is a major reprint of the original Challenge! (1991-1995) published in 1997. It contains the problems and full solutions to all Junior and Intermediate problems set in the Mathematics Challenge for Young Australians, exactly as they were proposed at the time. It is expanded to cover the years up to 1998, has more advanced typography and makes use of colour. It is highly recommended as a resource book for classes from Years 7 to 10 and also for students who wish to develop their problem-solving skills. Most of the problems are graded within to allow students to access an easier idea before developing through a few levels.

Challenge! 1999—2006 Book 2
JB Henry & PJ Taylor
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USSR Mathematical Olympiads 1989 – 1992
Edited by AM Slinko

Arkadii Slinko, now at the University of Auckland, was one of the leading figures of the USSR Mathematical Olympiad Committee during the last years before democratisation. This book brings together the problems and solutions of the last four years of the All-Union Mathematics Olympiads. Not only are the problems and solutions highly expository but the book is worth reading alone for the fascinating history of mathematics competitions to be found in the introduction.

Australian Mathematical Olympiads 1979 – 1995
H Lausch & PJ Taylor

This book is a complete collection of all Australian Mathematical Olympiad papers from the first competition in 1979. Solutions to all problems are included and in a number of cases alternative solutions are offered.

Chinese Mathematics Competitions and Olympiads Book 1 1981–1993
A Liu

This book contains the papers and solutions of two contests, the Chinese National High School Competition and the Chinese Mathematical Olympiad. China has an outstanding record in the IMO and this book contains the problems that were used in identifying the team candidates and selecting the Chinese team. The problems are meticulously constructed, many with distinctive flavour. They come in all levels of difficulty, from the relatively basic to the most challenging.

Asian Pacific Mathematics Olympiads 1989-2000
H Lausch & C Bosch-Giral

With innovative regulations and procedures, the APMO has become a model for regional competitions around the world where costs and logistics are serious considerations. This 159 page book reports the first twelve years of this competition, including sections on its early history, problems, solutions and statistics.

Polish and Austrian Mathematical Olympiads 1981–1995
ME Kuczma & E Windischbacher

Poland and Austria hold some of the strongest traditions of Mathematical Olympiads in Europe even holding a joint Olympiad of high quality. This book contains some of the best problems from the national Olympiads. All problems have two or more independent solutions, indicating their richness as mathematical problems.

Seeking Solutions
JC Burns

Professor John Burns, formerly Professor of Mathematics at the Royal Military College, Duntroon and Foundation Member of the Australian Mathematical Olympiad Committee, solves the problems of the 1988, 1989 and 1990 International Mathematical Olympiads. Unlike other books in which only complete solutions are given, John Burns describes the complete thought processes he went through when solving the problems from scratch. Written in an inimitable and sensitive style, this book is a must for a student planning on developing the ability to solve advanced mathematics problems.

101 Problems in Algebra from the Training of the USA IMO Team
Edited by T Andreescu & Z Feng

This book contains one hundred and one highly rated problems used in training and testing the USA International Mathematical Olympiad team. The problems are carefully graded, ranging from quite accessible towards quite challenging. The problems have been well developed and are highly recommended to any student aspiring to participate at National or International Mathematical Olympiads.

Hungary Israel Mathematics Competition
S Gueron

The Hungary Israel Mathematics Competition commenced in 1990 when diplomatic relations between the two countries were in their infancy. This 181-page book summarizes the first 12 years of the competition (1990 to 2001) and includes the problems and complete solutions. The book is directed at mathematics lovers, problem solving enthusiasts and students who wish to improve their competition skills. No special or advanced knowledge is required beyond that of the typical IMO contestant and the book includes a glossary explaining the terms and theorems which are not standard that have been used in the book.

Chinese Mathematics Competitions and Olympiads Book 2 1993–2001
A Liu
This book is a continuation of the earlier volume and covers the years 1993 to 2001.

Bulgarian Mathematics Competition 1992–2001
BJ Lazarov, JB Tabov, PJ Taylor & A Storozhev

The Bulgarian Mathematics Competition has become one of the most difficult and interesting competitions in the world. It is unique in structure combining mathematics and informatics problems in a multi-choice format. This book covers the first ten years of the competition complete with answers and solutions. Students of average ability and with an interest in the subject should be able to access this book and find a challenge.

International Mathematical Talent Search Part 1
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This 250-page book is aimed at advanced, senior students at Year 10 level and above.

Mathematical Contests – Australian Scene
Edited by PJ Brown, A Di Pasquale & K McAvaney

These books provide an annual record of the Australian Mathematical Olympiad Committee's identification, testing and selection procedures for the Australian team at each International Mathematical Olympiad. The books consist of the questions, solutions, results and statistics for: Australian Intermediate Mathematics Olympiad (formerly AMOC Intermediate Olympiad), AMOC Senior Mathematics Contest, Australian Mathematics Olympiad, Asian-Pacific Mathematics Olympiad, International Mathematical Olympiad, and Maths Challenge Stage of the Mathematical Challenge for Young Australians.

Mathematics Competitions
Edited by J Švrcek

This bi-annual journal is published by AMT Publishing on behalf of the World Federation of National Mathematics Competitions. It contains articles of interest to academics and teachers around the world who run mathematics competitions, including articles on actual competitions, results from competitions, and mathematical and historical articles which may be of interest to those associated with competitions.

Problems to Solve in Middle School Mathematics
B Henry, L Mottershead, A Edwards, J McIntosh, A Nakos, K Sims, A Thomas & G Vardaro

This collection of problems is designed for use with students in years 5 to 8. Each of the 65 problems is presented ready to be photocopied for classroom use. With each problem there are teacher's notes and fully worked solutions. Some problems have extension problems presented with the teacher's notes. The problems are arranged in topics (Number, Counting, Space and Number, Space, Measurement, Time, Logic) and are roughly in order of difficulty within each topic. There is a chart suggesting which problem-solving strategies could be used with each problem.

Teaching and Assessing Working Mathematically Book 1 & Book 2
Elena Stoyanova

These books present ready-to-use materials that challenge students understanding of mathematics. In exercises and short assessments, working mathematically processes are linked with curriculum content and problem solving strategies. The books contain complete solutions and are suitable for mathematically able students in Years 3 to 4 (Book 1) and Years 5 to 8 (Book 2).

A Mathematical Olympiad Primer
G Smith

This accessible text will enable enthusiastic students to enter the world of secondary school mathematics competitions with confidence. This is an ideal book for senior high school students who aspire to advance from school mathematics to solving olympiad-style problems. The author is the leader of the British IMO team.

ENRICHMENT STUDENT NOTES

The Enrichment Stage of the Mathematics Challenge for Young Australians (sponsored by the Dept of Innovation, Industry, Science and Research) contains formal course work as part of a structured, in-school program. The Student Notes are supplied to students enrolled in the program along with other materials provided to their teacher. We are making these Notes available as a text book to interested parties for whom the program is not available.

Newton Enrichment Student Notes
JB Henry

Recommended for mathematics students of about Year 5 and 6 as extension material. Topics include polyominoes, arithmetricks, polyhedra, patterns and divisibility.

Dirichlet Enrichment Student Notes
JB Henry

This series has chapters on some problem solving techniques, tessellations, base five arithmetic, pattern seeking, rates and number theory. It is designed for students in Years 6 or 7.

Euler Enrichment Student Notes
MW Evans & JB Henry

Recommended for mathematics students of about Year 7 as extension material. Topics include elementary number theory and geometry, counting, pigeonhole principle.

Gauss Enrichment Student Notes
MW Evans, JB Henry & AM Storozhev

Recommended for mathematics students of about Year 8 as extension material. Topics include Pythagoras theorem, Diophantine equations, counting, congruences.

Noether Enrichment Student Notes
AM Storozhev

Recommended for mathematics students of about Year 9 as extension material. Topics include number theory, sequences, inequalities, circle geometry.

Pólya Enrichment Student Notes
G Ball, K Hamann & AM Storozhev

Recommended for mathematics students of about Year 10 as extension material. Topics include polynomials, algebra, inequalities and geometry.

T-SHIRTS

T-shirts of the following six mathematicians are made of 100% cotton and are designed and printed in Australia. They come in white, Medium (Turing only) and XL.

Leonhard Euler T-shirt

The Leonhard Euler t-shirts depict a brightly coloured cartoon representation of Euler's famous Seven Bridges of Königsberg question.

Carl Friedrich Gauss T-shirt

The Carl Friedrich Gauss t-shirts celebrate Gauss' discovery of the construction of a 17-gon by straight edge and compass, depicted by a brightly coloured cartoon.

Emmy Noether T-shirt

The Emmy Noether t-shirts show a schematic representation of her work on algebraic structures in the form of a brightly coloured cartoon.

George Pólya T-shirt

George Pólya was one of the most significant mathematicians of the 20th century, both as a researcher, where he made many significant discoveries, and as a teacher and inspiration to others. This t-shirt features one of Pólya's most famous theorems, the Necklace Theorem, which he discovered while working on mathematical aspects of chemical structure.

Peter Gustav Lejeune Dirichlet T-shirt

Dirichlet formulated the Pigeonhole Principle, often known as Dirichlet's Principle, which states: "If there are p pigeons placed in h holes and $p>h$ then there must be at least one pigeonhole containing at least 2 pigeons." The t-shirt has a bright cartoon representation of this principle.

Alan Mathison Turing T-shirt

The Alan Mathison Turing t-shirt depicts a colourful design representing Turing's computing machines which were the first computers.

ORDERING

All the above publications are available from AMT Publishing and can be purchased online at:

www.amt.edu.au/amtpub.html or contact the following:

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The Australian Mathematics Trust

The Trust, of which the University of Canberra is Trustee, is a not-for-profit organisation whose mission is to enable students to achieve their full intellectual potential in mathematics. Its strengths are based upon:

- a network of dedicated mathematicians and teachers who work in a voluntary capacity supporting the activities of the Trust;
- the quality, freshness and variety of its questions in the Australian Mathematics Competition, the Mathematics Challenge for Young Australians, and other Trust contests;
- the production of valued, accessible mathematics materials;
- dedication to the concept of solidarity in education;
- credibility and acceptance by educationalists and the community in general whether locally, nationally or internationally; and
- a close association with the Australian Academy of Science and professional bodies.