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MATHEMATICS COMPETITIONS

JOURNAL OF THE

WORLD FEDERATION OF NATIONAL MATHEMATICS COMPETITIONS

AUSTRALIAN MATHS TRUST

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The views expressed here are those of the authors and do not necessarily represent the views of the Australian Mathematics Trust.

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From the President

Dear readers,

As I write this, preparation for our next mini-conference in Sydney, Australia in 2024 is starting in earnest, and I hope that many of you will be motivated to attend.

As most of you will be aware, our organisation was founded at the International Congress on Mathematical Education (ICME-5) in Adelaide in 1984. Now, the congress of the International Commission on Mathematical Instruction (ICMI) returns to Australia for ICME-15 from July 7th to 14th, 2024, and as has become a tradition, a WFNMC mini-conference is planned for the day before ICME begins, namely the 6th of July, 2024. As many people interested in mathematics competitions will be in Sydney at that time, it is hoped that this will be convenient for as many as possible.

You are cordially invited to submit a proposal for a talk to be presented at the mini-comference. Proposals should be approximately 50 to 100 words in length and include a title and the name and affiliation of the presenter. The lengths of the talks will depend on the time available, but will likely be 20-30 minutes.

Please send your proposals to the organising committee at **Krzysztof.Ciesielski@im.uj.edu.pl** using the subject line "WFNMC mini conference proposal". The deadlines for proposals is December 31st, 2023, with titles and abstracts due by March 31st, 2024.

If you are planning to attend ICME, Topic Study Group 2.3 with the title **Mathematics and creativity; mathematical competitions; mathematical challenge** will also be of interest. Information on how to submit proposals for inclusion in this TSG is available at https://icme15.org/icme-15-scientific-program/topic-study-groups/
tsg-2-3-mathematics-and-creativity-mathematical-competitions-mathematical-challenge/

Beside these meetings, we can also look forward to the upcoming publication of the book based on papers presented at our 2022 conference in Sofia, Bulgaria. This book, with the title **Engaging Young Students in Mathematics through Competitions, World Perspectives and Practices, Volume III Keeping Competition Mathematics Engaging in Pandemic Times**, published by World Scientific Publishing, is on track to appear by the end of 2023.

Also, as always, you are invited to submit papers for publication in the journal! Your active participation in all activities of our organisation is greatly appreciated, and I hope to see a great number of you in Sydney!

Robert Geretschläger

Editor's Page

Dear Competitions enthusiasts, readers of our *Mathematics Competitions* journal!

Mathematics Competitions is the right place for you to publish and read the different activities about competitions in Mathematics from around the world. For those of us who have spent a great part of our life encouraging students to enjoy mathematics and the different challenges surrounding its study and development, the journal can offer a platform to exhibit our results as well as a place to find new inspiration in the ways others have motivated young students to explore and learn mathematics through competitions. In a way, this learning from others is one of the better benefits of the competitions environment.

Following the example of previous editors, I invite you to submit to our journal *Mathematics Competitions* your creative essays on a variety of topics related to creating original problems, working with students and teachers, organizing and running mathematics competitions, historical and philosophical views on mathematics and closely related fields, and even your original literary works related to mathematics.

Just be original, creative, and inspirational. Share your ideas, problems, conjectures, and solutions with all your colleagues by publishing them here. We have formalized the submission format to establish uniformity in our journal.

Submission Format
FORMAT: should be LaTeX, TeX, or for only text articles in Microsoft Word, accompanied by another copy in pdf. However, the authors are strongly recommended to send article in TeX or LaTeX format. This is because the whole journal will be compiled in LaTex. Thus your Word document will be typeset again. Texts in Word, if sent, should mainly contain non-mathematical text and any images used should be sent separately.
ILLUSTRATIONS: must be inserted at about the correct place of the text of your submission in one of the following formats: jpeg, pdf, tiff, eps, or mp. Your illustration will not be redrawn. Resolution of your illustrations must be at least 300 dpi, or, preferably, done as vector illustrations. If a text is embedded in illustrations, use a font from the Times New Roman family in 11 pt.
START: with the title centered in Large format (roughly 14 pt), followed on the next line by the author(s)' name(s) in italic 12 pt.
MAIN TEXT: Use a font from the Times New Roman family or 12 pt in LaTex.
END: with your name-address-email and your website (if applicable).
INCLUDE: your high resolution small photo and a concise professional summary of your works and titles.
Please submit your manuscripts to María Elizabeth Losada at
`director.olimpiadas@uan.edu.co`

We are counting on receiving your contributions, informative, inspired and creative. Best wishes,

Maria Elizabeth Losada
EDITOR

Some Reminiscence of K. P. Shum

Andy Liu

K. P. and I attended the same high school in Hong Kong and the same University in Canada. However, we were never together on either campus during our student days. He had left before I came in.

We met later at the University of Alberta. He was taking a sabbatical from the Chinese University of Hong Kong, and decided to come back to his alma mater. He was scheduled to teach a class, but a minor traffic accident in Hong Kong delayed his departure.

If our Department of Mathematics were a baseball team, I would be the designated hitter. I taught his class until he was able to take over. Casual conversation revealed our intertwined academic lineage. We renewed our acquaintance several times during my various visits to Hong Kong, but the key moment came in 1994. He succeeded in bringing the International Mathematical Olympiad to Hong Kong. I had been involved in the movement since 1981. So he summoned me to be his designated hitter.

He wished to host the I.M.O. at the Chinese University of Hong Kong. However, there was a hostile force which did not wish the event to take place there. K. P. was not to be driven from his home base, and the die was cast.

The hostile force made things difficult for the team of workers under K. P., setting the Chair of the Department of Mathematics against us. K. P. got the Dean of Science to clamp down on the Chair. The hostile force got the President of the University to clamp down on the Dean. K. P. went to the Chancellor of the University, the major financial backer of the institution, to clamp down on the President.

It did not really help, because it took too long for this chain reaction to have any effect. We were limited to the use of a single office with one photo-copier. It was really fortunate that the overheated machine did not conk out until the day after we had printed all the question papers. All subsequent printings were done on the high street at K. P.'s expense.

Because of the slow progress, we were busy stuffing envelopes at 2:30 am on the day when the contest would start at 8:30 am! We had to check that the paper(s) of the correct language(s) were inside individual envelopes. We seemed to be at the brink of disaster almost every moment, but somehow we managed to stay afloat.

Some years passed. Then I was jolted by an announcement that K. P. had passed away. This was soon rescinded, as K. P. had made a miraculous recovery. I was told that he had already been placed on the stone table in the morgue. Suddenly, he sat up, almost scaring the attendant to death. He was in a wheel-chair for about half a year, but a year later, he was flying to distant places to attend conferences. When I finally caught up with him, he said that he was working on a graph theory problem while lying on the stone table, figuring out how to get around certain blocking vertices. When he finally solved the problem, his brain took lessons and worked around the blockage within, bringing him back to life.

I met him again in 2016, when he brought the I.M.O. to Hong Kong for a second time. Again, I was summoned. However, things went a lot more smoothly this time. The host was the Hong Kong University of Science and Technology which was completely behind the event. I was able to enjoy my stay without doing any work.

I last met K. P. in Hong Kong in 2017, when he was presented with the Erdős Award from the World Federation of National Mathematics Competitions, a well-deserved recognition of his many contributions in the Olympiad scene.

When I heard that K. P. had passed away this time, I was sure that a miracle was in the offing. Unfortunately, his time had finally come. He was a kind man and a great leader. I am proud to be among his followers.

Andy Liu
acfliu@gmail.com

The Main Dickstein Prize for Marcin E. Kuczma

The Polish Mathematical Society awards 3 Main Prizes. They are the Main Stefan Banach Prize that is given annually for results in pure mathematics, the Main Hugo Steinhaus Prize that is given annually for results in applications of mathematics, and the Main Samuel Dickstein Prize, that is awarded once every two years. The Dickstein Prize is given for outstanding achievements in mathematical culture, in particular concerning the history of mathematics, mathematical education and the popularisation of mathematics. The first laureate of the Dickstein Prize was in 1979 the first President of the Main Committe of the Polish Mathematical Olympiad, Stefan Straszewicz.

In 2023 the Dickstein Prize was given to Marcin Emil Kuczma from Warsaw University. Kuczma received this award for more than 50 years of activity in the field of mathematical education and popularisation of mathematics with particular emphasis on more than 40 years of editing "Club 44 M" column in the Polish popular monthly *Delta* and nearly 50 years of gigantic work for the Mathematical Olympiad, as well as other publications and his activity concerning the popularising of mathematics.

In 1981, Klub 44 (Club 44, now Club 44 M) was established in the monthly *Delta*. It is a non-standard mathematical competition – a kind of a league for readers; problems are announced in each issue, then readers may send their solutions to the editor, and all submitted solutions are evaluated. Readers have almost three months to solve the problems, so it is easy to guess that those problems cannot be easy. The column was initiated by Marcin E. Kuczma and since the very beginning it has been edited by him – Kuczma presents the problems (he is the author of many of them) and evaluates the submitted solutions.

The league is very popular and makes an excellent contribution to popularising mathematics and non-standard problems. Problems presented there are also being solved by people who do not send solutions to the editor. Among people taking part in the league are school students, university students, PhD students, teachers, scientists, and people whose job is currently not associated with mathematics. The problems posted in the column are sometimes extremely ingenious. The regular evaluation of the solutions is a huge job. It was for his running of Club 44 column that M.E.Kuczma was awarded the Hilbert Medal in 1992 by the World Federation of National Mathematics Competitions (now, after the death of Pál Erdős, the Erdős Medal is awarded).

Marcin E. Kuczma has been a member of the Regional Committee of the Polish Mathematical Olympiad in Warsaw since 1965, of which he served as the secretary of the Committee for 29 years. In addition, he was a member of the Main Committee of the Polish Mathematical Olympiad for 26 years. If we count the years in which he was a member of both Committees doubly (they are two Committees with significantly different duties), this calculates to 74 years. He served as the Scientific Secretary of the Main Committee for 8 years. For many years he played a crucial role in creating olympiad problems. Many of the problems posed at the competition were of his authorship. When the work of the Main Committee was reorganised in 1996 and a special "Problem Commission" responsible for the preparation of the olympiad problems was established, M.E.Kuczma became its first chairman and served in this capacity for 6 years.

Marcin E. Kuczma served as a member of jury at the International Mathematical Olympiad, the Austrian-Polish Mathematical Competition, and the Baltic Way Mathematical Competition. Several times he was a member of IMO Problems Selection Committee. He was present at 23 (!) IMO competitions: as a contestant (twice; in 1962 his score put him in the fifth position), as the deputy leader (once), as the team leader (6 times), as a PSC-member plus coordinator (8 times), just as a coordinator (5 times) and a honorary guest (once).

Walter E. Mientka (1925–2014), whose name and achievements are very well known to members of WFNMC, wrote that Marcin E. Kuczma "is identified throughout the world of mathematics as a research mathematician, creator of intriguing problems and as one who possesses unusual mathematical ability".

Marcin Emil Kuczma is the author of four excellent books containing olympiad problems – one in Polish, three in English. For many years, volumes *Problems from Mathematical Olympiads* were published in Poland. In these books were presented the problems from the Olympiads of successive five-year periods and the international competitions of those years, and – perhaps above all – the full solutions to these problems, accurately and carefully worked out. The author of the first volumes was Stefan Straszewicz. Volume 8 (*Olimpiady Matematyczne*, vol. 8, Wydawnictwa Szkolne i Pedagogiczne, Warszawa 2000) was written by M.E.Kuczma. Moreover, for many years, special volumes "Reports of the n^{th} Mathematical Olympiad" were published each year in Poland. In these reports, first of all, the problems and solutions were contained. "Problems and solutions" parts in fourteen volumes of "Reports" were prepared and edited by M.E.Kuczma. His books published abroad are: *Polish and Austrian Mathematical Olympiads 1981–1995*, *Selected Problems with Multiple Solutions* (coauthor: Erich Windischbacher), An Australian Mathematics Trust Publication, Canberra 1998; *144 Problems of the Austrian-Polish Mathematics Competitions (1978–1983)*, The Academic Distribution Center, Freeland 1994; *International Mathematical Olympiads 1986–1999*, The Mathematical Association of America, Washington 2003.

The problems proposed by Marcin E. Kuczma have been also published in “Problem corners” of many prestigious international journals, in particular: *American Mathematical Monthly*, *Elemente der Mathematik*, *Crux Mathematicorum*, *Journal of Recreational Mathematics*.

Marcin E. Kuczma has been actively involved in the work of WFNMC. He was the organiser of one of the main sections of the 3$^{\text{rd}}$ WFNMC Congress (Zhong Shan, 1998) entitled *Proposal of Competition Problems*, and he gave an invited lecture at the 8$^{\text{th}}$ Congress (Graz, 2018).

Marcin E. Kuczma’s work in popularising mathematics is not limited to the subject of problems and competitions. For many years, the series “Biblioteczka Matematyczna” (“A Small Mathematical Library”) series was published in Poland, and several volumes written by famous Polish and foreign mathematicians, including Wacław Sierpiński and Zdzisław Opial, were published there. M.E.Kuczma is the author of a fascinating book in this series entitled *O szeregach liczbowych* (*On Numerical Series*), vol. 40 in “Biblioteczka Matematyczna”. He is also an author of interesting popular articles published in the monthly *Delta* and in the journal of the Polish Centre for Mathematical Culture: *Mathematics, Society, Teaching*. The article *The History of the World Measured by Factorials*, published in *Delta* in 2009, was honoured with the so-called Deans’ Award as the best article published in *Delta* in the academic year 2008/2009. In the 21 years of this award’s existence, it has been only six times awarded for mathematical articles.

Another M.E.Kuczma’s important activity for mathematics education is his translation work. He translated into English four important mathematical monographs by Polish authors: Włodzimierz Mlak, *Hilbert spaces and operator theory*; Zofia Szmydt, *Fourier transformation and linear differential equations*; Wiesław Szlenk, *An introduction to the theory of smooth dynamical systems*, and Wiesław Żelazko, *Banach algebras*. It is very important that these monographs have English versions and that they are very well translated – by a competent mathematician.

In the decision of awarding the Dickstein Prize, strictly scientific achievements are not taken into account. Nevertheless, in presenting the laureate’s profile it should be noted that Marcin E. Kuczma is the author of several research papers, reviewed in the reference databases *Mathematical Reviews* and *Zentralblatt MATH*.

Congratulations, Marcin.

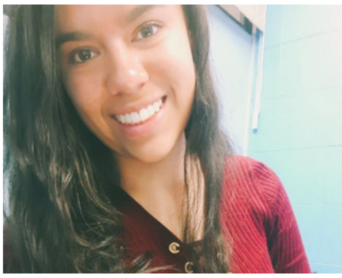
Krzysztof Ciesielski

Mathematics Olympiads Curriculum for Primary School in Ibero-America

Lizbeth Alvarado, Luis Cáceres and Ariana Rodríguez

Luis Caceres is a professor of mathematics at the University of Puerto Rico, Mayaguez Campus. He is also the founder and director of the Puerto Rico Mathematics Olympiads. He holds a Ph.D. in mathematical logic from the University of Iowa in the United States. He has worked with the Math Olympiads for the past two decades and has collaborated on different educational and math competition projects in various countries. He received the Paul Erdos Award from the WFNMC in 2016. He is vice president of the Association "Kangourou sans Frontieres", which organizes the largest international mathematical competition in the world.

Ariana Rodríguez obtained her bachelor's degree in Teaching Mathematics with Technological Environments at the Technological Institute of Costa Rica in 2021. That same year she entered a master's degree in Teaching Mathematics at the Secondary Level at the University of Puerto Rico, Mayaguez campus. She is interested in continuing with her doctoral studies and researching topics related to the Mathematics Olympiads, teaching techniques with the aim of improving the academic performance of students using current technologies, among other topics related to the teaching of mathematics.



Lizbeth Alvarado graduated from the Technological Institute of Costa Rica in Teaching Mathematics with Technological Environments. She is a master of science student at the University of Puerto Rico, Mayagüez Campus in the Mathematics Teaching Program at the Secondary Level. Her main interests include problem solving, mathematics didactics and the inclusion of technologies in the teaching process.

Abstract
This article investigates the curriculum and main topics included in the mathematical Olympiads tests for primary schools on several Ibero-American countries. The purpose of this research is to structure a topic guide for students who wish to participate in future competitions. To obtain this information, an analysis of each exam exercise was conducted to classify each problem into a specific area of mathematics and the predominant theme of the area. Finally, the information extracted from the analysis allowed us to propose a guide to the specific mathematical areas and topics covered on the tests, as well as provide examples to teachers, parents and students for test preparation.

Introduction

The objective of this article is to determine the predominant areas and topics integrated into the primary mathematics Olympiads in some Ibero-American countries. To achieve this, first, objectives and other competition characteristics of the primary mathematical olympiads are discussed according to the country in which they take place. Next, the frequency of each mathematical area was determined according to the number of exercises each area includes, some exercises found whose resolution focuses on a specific mathematical area are also exemplified, all of the above in order to generate a guide that allows orienting the preparation of the participating students towards the areas or themes that are more frequent in the competitions, so that they can have a clearer picture and obtain a better performance in them.

In this research a qualitative methodology is employed; it covers a collection of data through electronic sources, articles, among others; that contribute to the description and understanding of the project. A non-exhaustive search of different exams applied in recent years in mathematics Olympiads in Ibero-American countries was carried out. Then the exercises of each one of them were analyzed and classified by areas of mathematics: number theory, geometry, algebra (in this one the topic of patterns was taken as an external section, since being primary competitions there are quite a few exercises , which are solved with the use of patterns), arithmetic, counting, logic, statistics and probability, and at the same time a specific topic or skill that a student could apply to solve the exercise was identified.

Mathematics Olympiads for Primary School in Ibero-America

The Mathematics Olympiads in elementary school are educational competitions based on the resolution of timed mathematical problems among students who are attending elementary or primary education, i.e., students between the ages of 6 and 12 approximately. The competitions are held by countries, regions, or even between students from different parts of the world. One of the objectives of these competitions is to enhance the students' skills in mathematical thinking, problem solving, strategy development, teamwork and working under pressure, among others. Also, as mentioned in [19], these skills support math teachers, since they help stimulate the skills of talented students.

Even though these Olympiads are held in different countries around the world, this paper only focuses on mathematical Olympiads exams administered in Ibero-American countries, such as Puerto Rico, Costa Rica, Argentina, Colombia, Mexico, Spain, Bolivia, Nicaragua, El Salvador, Paraguay, Ecuador, Portugal, and some classified as International. Exams for students from fourth to sixth grade of primary school, that is, between 9 and 12 years of age, approximately are mostly considered. In addition, it is important to mention that since these are different countries, the number of exercises and their difficulty is not the same in all the exams. For instance, in Puerto Rico, the exam consists of 20 single-choice questions with different levels of difficulty, while in Portugal, the exam consists of four essay questions, all with a high level of difficulty.

Analysis of the number of exercises per area of Mathematics

Before addressing the most predominant areas of mathematics in Mathematics Olympiad exercises, it is important to emphasize that most of the registration pages or brochures for said competitions do not contain an agenda or content guidelines for students and parents. However, there are some articles in which certain predominant topics are mentioned in broad strokes or in competitions such as the ”Mathematical Kangaroo” that has made an effort to indicate the curriculum of its competition. In these articles and competitions the predominant topics are geometry, number theory, algebra and combinatorics. [8] [1]

After evaluating exercises from different recent exams (2018-2022) offered in some Ibero-American countries, the predominant topics include logic, statistics, patterns (although considered an algebra topic, patterns are frequently included in primary mathematics Olympiads tests, so it was analyzed separately) and arithmetic. A total of 236 exercises were reviewed, which were classified by area of mathematics, as shown in Table 1 below.

For each of the mathematical areas indicated in Table 1, the most frequent specific topics are indicated below:

Arithmetic

- Basic operations.
- Combined operations (mathematical expressions that involve two or more operations such as addition, subtraction, multiplication, and/or division).
- Place value in base ten.
- Unit conversion (volume, time, mass, length).
- Percentages.

Geometry

- Area and perimeter of geometric figures (squares, rectangles, circles, triangles).
- Hatched area.
- Measurement of internal angles of polygons (triangles, rectangles and squares).
- Similarity of triangles.
- Geometric transformations (rotation, reflection, translation).
- Location of points in the cartesian plane.

Logic

- Case-based problem-solving.
- Object classification.
- Puzzle completion.

Number Theory

- Even, odd and prime numbers.
- Decomposition into prime numbers.
- Divisibility by 2, 3, 4, 5, 9, 10 and number of divisors of a number.
- Division algorithm.
- Multiples of 2, 3, 4, 5, 9 and 10.
- Least common multiple.

Algebra

- Linear equations.
- Proportions.
- Sum of the first n natural numbers (Gaussian sum).
- Patterns.

Counting

- Combinations and permutations making a list of possible cases.
- Trees.

Statistics

- Measures of central tendency (mean, mode and median).

Mathematics Areas	Percentage of Exercises
Arithmetic	25.85%
Geometry	23.31%
Logic	20.34%
Counting	8.05%
Number Theory	7.20%
Algebra	7.20%
Patterns	5.08%
Statistic and probability	2.97%

Table 1: Percentage of exercises according to their area of mathematics.

Next, the characteristics of each area are detailed, along with examples.

Arithmetic

The exercises classified in the area of arithmetic are mostly solved through the use of basic operations, combined operations, or place value in base 10.

Exercise 1
XXXI Argentinian Mathematics Olympiad Ñandú Second Level (2022)

Juana needed to assemble fruit plates using apples and apricots. She cut each apple into 8 pieces and each apricot into 6 pieces. On each plate, she placed 6 pieces of apple and 4 pieces of apricot. Using all the apples and all the apricots that Juana had, she was able to put together 24 fruit plates. How many apples and how many apricots did Juana have?

Answer:
Since Juana was able to assemble 24 fruit plates, she had a total of $24 \cdot 6 = 144$ apple pieces (each plate contained 6 apple pieces) and $24 \cdot 4 = 96$ apricot pieces (each plate contained 4 apricot pieces). From this, since she cut each apple into 8 pieces and she had 144 in total, Juana had $144 \div 8 = 18$ apples, and since she cut each apricot into 6 pieces, she had $96 \div 6 = 16$ apricots.

Geometry

The geometry area was the second most predominant in the analyzed exercises, and involves topics, such as the calculation of perimeters, volumes and areas, rotations and translations of different geometric figures, distance between points in the Cartesian plane and angle measurements.

Exercise 26
35ª Mexican Mathematics Olympiad (2021)

A line goes zig-zag between ends A and B of the diameter of a circumference touching points on the circumference as shown in the diagram. If, apart from A and B, it touched the circumference exactly 4 times, what is the measure of the angle α?
(a) 60° (b) 72° (c) 75° (d) 80° (e)Other answer

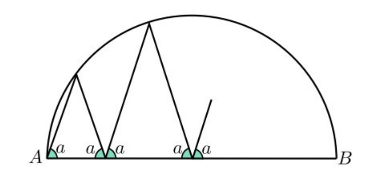


Figure 1: Exercise 26. Mexican Mathematics Olympiad (2021)

Answer:
Let's reflect the figure across its diameter.

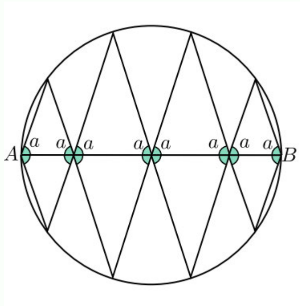


Figure 2: Answer 26. Mexican Mathematics Olympiad (2021)
We have parallel lines that form the same angles, so all arcs are equal. The angle sought is half the interior angle in a regular decagon, that is, $\frac{8\cdot180}{2\cdot10} = 72°$. The answer is (b).

Logic

This section includes exercises that do not necessarily involve the use of any specific formula or topic, but are based solely on the use of mathematics reasoning. Some exercise examples with this peculiarity include completing figures or puzzles, placing or rearranging objects that meet a given characteristic, finding figures within other figures, and forming figures from connecting objects.

Exercise 4
Portuguese Mathematics Olympiad (2021)

At the end of a Mathematics Olympiad, Mariana and Sofía were talking. Looking around, Mariana noticed that the other 8 participants were also talking in pairs. Mariana then asked each participant how many participants she had talked to and got 9 different answers. What was Sofia's response?

Answer:

In total there were 10 participants, hence, the largest of the possible answers given to Mariana is 9, since no participant could speak with more than 9 people, then as all the answers were different, the answers given to Mariana were 1, 2, 3, 4, 5, 6, 7, 8 and 9. Sofía did not give 1 as an answer, because whoever answered that she was talking with a participant necessarily had to talk with whoever gave 9 as an answer and it is known that Sofía talked with Mariana, therefore, her answer cannot be 1. Sofía did not give a 2 as an answer either, since the person who answered that she talked with two participants necessarily had to talk with the one who gave the answer 9 and with the one who gave the answer 8 and since Sofía talked with Mariana, it cannot be that she answered 2. In the same way, Sofía could not give 3 or 4 as an answer. Therefore, Sofía's answer was that she talked with 5 participants, who answered 6, 7, 8, 9 and Mariana.

Counting

This section includes exercises in which the student must determine the number of ways an event can occur. If they follow a specific order or repetition, they must be able to determine the number of possible cases without necessarily having to describe them one by one.

Exercise 6
Tenth Colombian Regional Olympiads of Mathematics in Primary School (2021)

Santiago's grandfather buys raffle tickets every day. He always chooses the number that corresponds to the sum of the digits in the date. For example on August 26, 2021, (08/26/2021) he bought the number $21 = 0 + 8 + 2 + 6 + 2 + 0 + 2 + 1$. How many times did he buy or will he buy the number 12 during the year 2021?

(a) 29 (b) 15 (c) 30 (d) 12 (e) It is not known

Answer:

As the tickets were purchased during the year 2021 and the digits for this year add up to 5, then the sum of the digits for the day and the month in which it is must be 7 to buy the number 12. Therefore, when it is the month 01 or month 10, the sum of the digits of the day must be 6 (06, 24 and 15), from which 6 possible dates are obtained. Similarly, when you are in month 02 or month 11, the sum of the day's digits must be 5, giving 6 possible dates. For the month 03 and 12 there are then 8 dates; for the month 04 four dates; for month 05 three dates and for month 06 two dates are obtained. The months 07, 08 and 09 are not being considered account because the sum of their digits would be equal to or greater than 7. Therefore, the correct answer is 29 possible dates.

Number Theory

In this category, problems whose solution is based on the use of divisibility properties were analyzed, such as criteria for divisibility by 2, 3, 4, 6, 9 and 10; number of divisors of a number; and multiples of 2, 3, 4, 5, 6, 9, 10. In some cases the division algorithm is used to determine remainders, also the factorization of integers as the product of prime numbers, as well as, the notion of even, odd and prime numbers must be known.

Exercise 14
National Mathematics Olympiad El Salvador - 2021 (Sixth grade)

Determine how many positive integers divide $5^8 + 2 \times 5^9$.

Answer:
Let us consider the prime decomposition of the number $N = 5^8 + 2 \times 5^9$. For this we note that:

$$N = 5^8(1 + 2 \times 5) = 5^8 \times 11$$

Each divisor of N is formed by considering a power of 5 and a power of 11. There are 9 options for the 5th power (0 to 8) and 2 options for the 11th power (0 and 1). So there are $9 \times 2 = 18$ divisors. The following table presents all the dividers:

$5^0 = 1$	5^1	5^2	5^3	5^4	5^5	5^6	5^7	5^8
$5^0 \times 11$	$5^1 \times 11$	$5^2 \times 11$	$5^3 \times 11$	$5^4 \times 11$	$5^5 \times 11$	$5^6 \times 11$	$5^7 \times 11$	$5^8 \times 11$

The answer is 18 positive integer divisors.

Algebra

Exercise 5
International Mathematics Olympiad for Primary Education (2020)

From the information in the image, how much does Rex weigh?

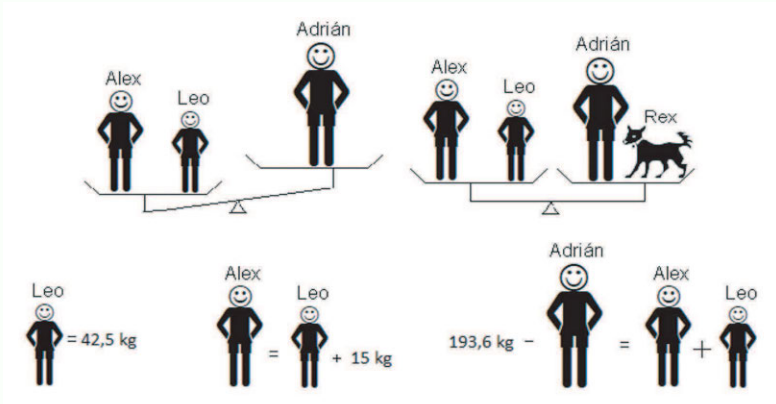


Figure 3: Exercise 5: OLIMPRI (2020)

Answer:
From the information, Alex's weight is $42.5 + 15 = 57.5$ kg. So Alex and Leo together weigh $57.5 + 42.5 = 100$ kg. Therefore, Adrian's weight is $193.6 - 100 = 93.6$ kg. Hence, since Alex and Leo weigh the same as Adrian and Rex, then Adrian and Rex must weigh 100 kg, and since Adrian weighs 93.6 kg, then Rex weighs $100 - 93.6 = 6.4$ kg.

In this section, the exercises that require the use of symbols or letters to represent a quantity were taken into account in order to solve them, although at the elementary level sometimes the use of algebraic equations as such has not been delved into, students have the notion of the same, to be able to solve exercises.

In addition, patterns have been placed as a specific section of Algebra because at the elementary level, a large number of exercises in this area refer to the use of patterns or sequences.

Patterns

Exercise 5
National Mathematics Olympiad El Salvador (2021)

The natural numbers starting from the number 3 are shown in the following figure:

	C_1	C_2	C_3	C_4	C_5	$\cdots$
F_1	3					
F_2	4	5				
F_3	6	7	8			
F_4	9	10	11	12		
F_5	13	14	15	16	17	
$\vdots$						

Figure 4: Exercise 5: ONM El Salvador (2021)

Determine the row and column where the number 2021 is located.

Answer:

A pattern can be observed in the final numbers of each row, these numbers can be obtained from the formula $a_n = \dfrac{n(n+1)}{2} + 2$, where n represents the row number. The formula was obtained based on the pattern these numbers follow from the Gauss sum. Substituting n for 63, the final number on row 63 is 2018, then, the number 2021 will be found in row 64, column 3.

Statistics and probability

Exercise 23
35ª Hummingbird Mathematical Olympiad - Costa Rica (2021)

A wooden box contains 15 gray balls, 25 blue balls, 32 white balls, 12 red balls and 28 green balls. What is the approximate probability of drawing a red ball?

(a) 0.29 (b) 0.11 (c) 0.22 (d) 0.25

Answer:

Note that the total number of balls in the box is 112, so the probability of drawing a red ball is $\dfrac{12}{112} \approx 0.11$. Therefore, the correct answer is (b).

Problems that require measures of central tendency or calculation of probabilities to solve them were classified in this area. However, the number of problems found in these areas was lower, and those that were found belong to exercises from Costa Rica and Nicaragua. In the other countries, no exercise involving statistics or probability was found.

Interdisciplinarity in the proposed exercises

Previously, problems classified in specific areas were studied. However, not all exercises can be classified within a single idea or theme, requiring different ideas or concepts outside the selected area or theme to obtain a solution. Hence, the following exercise is an example that can be classified in more than one area.

Exercise 39
35ª Mexican Mathematics Olympiad (2021)
Geometry and Algebra

The length of one of the sides of a rectangular garden increased by 20% and the length of the other side increased by 50%. Hence, the resulting garden was in the form of a square, as shown in the diagram. If the shaded area is $30m^2$, what is the area of the original rectangle?

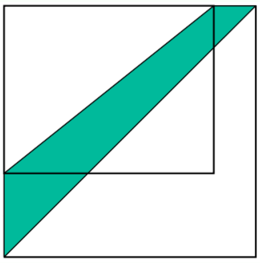


Figure 5: Exercise 39. 35ª Mexican Mathematics Olympiad (2021)

(a) $60m^2$ (b) $65m^2$ (c) $70m^2$ (d) $75m^2$ (d) $80m^2$

Answer:
Suppose the side of the original rectangle that was increased by 20% measures x. Hence, the square is $1.2x$ on one side, and $0.8x$ on the other side. Let us divide the shaded region into two triangles, as indicated in the diagram.

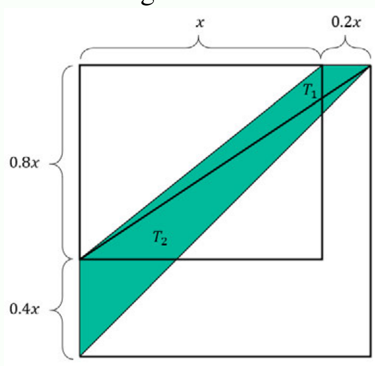


Figure 6: Exercise 39. 35ª Mexican Mathematics Olympiad (2021)

o the shaded area is

$$30 = \frac{0.2x \cdot 0.8x}{2} + \frac{0.4x \cdot 1.2x}{2} = \frac{0.16x^2 + 0.48x^2}{2} = 0.32x^2$$

Hence, $x^2 = 93.75$.

So the area of the original rectangle is $x \cdot 0.8x = 0.8x^2 = 0.8 \cdot 93.75 = 75m^2$. Therefore, the correct answer is option (d).

Conclusions and Future Work

In this work, an explicit list of predominant topics in the primary mathematics Olympiads tests from Ibero-America is provided. Although, as can be seen from the information above, in all the countries the exams have different approaches both in structure, content and length, they all share some objectives, such as fostering and developing mathematical thinking skills, problem solving, and strategy development. For this reason, it is important for the student to have access to a guide to help them prepare for these tests. That is, having a document that tells students the topics they should be more focused on and the most important strategies they should know. This will allow them to better prepare for, and feel more confident when taking the test.

That is why, based on the collected information[1], a brochure should be made available in which each of the topics mentioned in this writing is explained and examples of exercises that appear in different mathematics Olympiad tests are addressed for each one of these areas. Participants will benefit while preparing for these competitions since they will have a guide to follow.

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Miss Perkins's Quilt

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At the time of writing, both Yongye Lai and Hanqing Liu were grade eight students in Shenzhen Middle School. Now each is a first year university student.

"Mrs. Perkins's Quilt and Other Square-Packing Problems" is the title of Martin Gardner's September 1966 column of *Mathematical Games* in *Scientific American*. The column is later anthologized in [4]. It was based on an article by John Conway [1]. There was also a follow-up article [5]. Mrs. Perkins was a ficitious character introduced by Henry Dudeney in [2] and [3]. The main problem of Conway's article is dissecting a square with integer sides into the fewest smaller squares which need not be different or all different. We have found more details in a secret document, containing a problem which is related to but nevertheless independent of the original one.

It turned out that in her will, Mrs. Perkins left her $k \times k$ quilt to her niece. Miss Perkins was annoyed to discover that a corner of the quilt had worn off. She was able to remove the frailed edges by cutting off a 1×1 square at that corner. Now she wished to dissect the remaining part of the quilt into as few squares as possible. She denoted this minimum number by $p(k)$.

Clearly, $p(1) = 0$, and it was easy to verify that $p(2) = 3$, $p(3) = 5$, $p(4) = 6$ and $p(5) = 7$. Figure 1 showed constructions which attained these values.

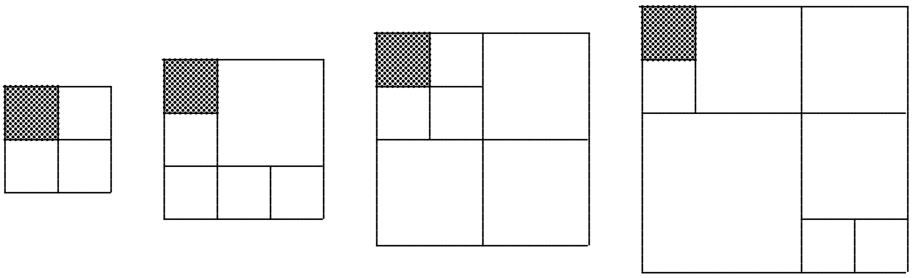

Figure 1

The constructions for $k = 2$ and $k = 4$ could be generalized to show that $p(2^t) \leq 3t$. The constructions for $k = 2$, $k = 3$ and $k = 5$ could be generalized to $p(F_t) \leq 2t - 3$, where $\{F_t\}$ denoted the Fibonacci sequence defined by $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. The next case for both sequences was $k = 8$, as shown in Figure 2, where $p(2^3) = 9 = p(F_6)$.

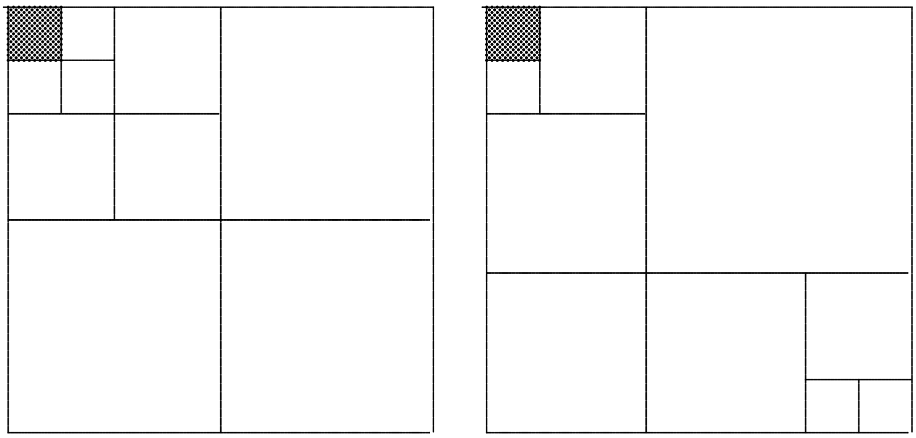

Figure 2

Miss Perkins believed that the new problem was just as diffucult as her aunt's original problem. She had no idea about how to approach it in general. She consulted the computer wizard **George Sicherman** of Red Bank, New Jersey. He provided lower bounds for $p(k)$, $2 \leq k \leq 25$, along with diagrams, including Figure 3 which shows alternative constructions for validating the upper bounds she had stated for the special cases $p(2^4) = p(16) \leq 12$ and $p(F_7) = p(13) \leq 11$.

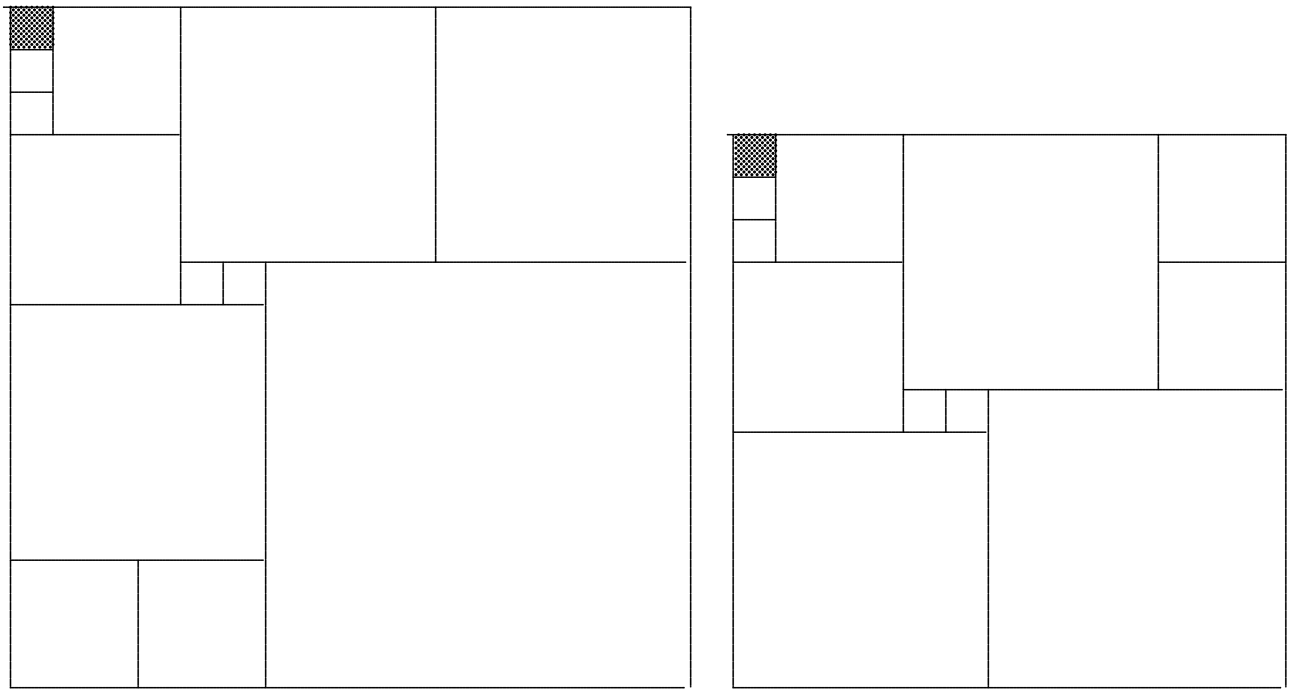

Figure 3

She copied the lower bounds for $17 \leq k \leq 25$ in the table below. She noted in particular that $p(F_8) = p(21) \leq 13$.

$k =$	17	18	19	20	21	22	23	24	25
$p(k) \leq$	12	12	12	12	13	13	13	13	13

So far, the function $p(k)$ behaved monotonically. However, she did not believe that this would persist, though a counter-example was nowhere in sight. She felt that she had simply offered another unsolved problem to the mathematical community.

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Properties of k-Descending Trees

Agniv Sarkar and Eric Severson

Agniv Sarkar is a current high schooler at Proof School, San Francisco, graduating in 2025. He did this work during his freshman year and has also researched monotonic functions and random processes. He is currently studying the decidability of problems involving matrices and researching trichoplax adaherans. Outside of school, he is a competitive fencer at the national level.

Eric Severson is currently teaching math and computer science at Proof School in San Francisco. He has a PhD in Applied Mathematics at UC Davis, working with professor David Doty. His research was at the intersection of distributed computing and molecular computing, focusing on the related models of population protocols and chemical reaction networks. He enjoys making nice math visuals. Lately making a lot of interactive objects with Geogebra. He is also learning the Manim animation library and made one video for a contest last summer.

Abstract
This was research presented at the Worldwide Federation of National Math Competitions in Bulgaria during 2022.
For any real valued $k > 1$, we consider the tree rooted at 0, where each positive integer n has parent $\lfloor \frac{n}{k} \rfloor$. The average number of children per node is k, thus this definition gives a natural way to extend k-ary trees to irrational k. We focus on the sequence r_d: the count of nodes at depth d. We first prove there exists some constant $\rho(k)$ such that $r_d \sim \rho(k) \cdot k^d$. We then study a family of values $k = \frac{a+\sqrt{a+4b}}{2}$, where we prove the sequence satisfies the exact recurrence $r_d = a \cdot r_{d-1} + b \cdot r_{d-2}$. This generalizes a special case when k is the golden ratio and r_d is the Fibonacci sequence.

Introduction

This contains a piece of original research centered within graph and number theory with a surprising connection to the Josephus Problem. The research was carried out by myself (Agniv Sarkar) and my mentor (Eric Severson) throughout the high school year of 2021-2022.

This research was done to observe patterns seen in the ϕ-tree generated with this section definition. This became a very nice number theoretic problem, and when we began to look at the asymptotics of these trees, we found that there was a connection to the Josephus problem and calculating the solution to the problem.

The trees themselves are most similar to a k-ary tree with this section definition.

Definition. A k-ary tree is a rooted tree such that each node has no more than k children. A complete k-ary tree is a rooted tree such that each node has exactly k children.

This tree is commonly used as a data structure in computer science, such as through a Binary Search Tree, or a 2-ary tree. However, the k in the definition does not generalize nicely to non integer k, and that is where [3] defines a "rhythmic tree," which is this section definition.

Definition. Let $p, q \in \mathbb{Z}$ such that $p > q \geq 1$. Then,

- Rhythm of directing parameter (q, p) is a q-tuple r non-negative integers whose sum is p.

$$r = (r_0, r_1, \ldots, r_{q-1}), \text{ and, } \sum_{i=0}^{q-1} r_i = p.$$

- A rhythm r is valid if it also satisfies

$$\forall k \in \{0, 1, \ldots, q-1\}, \sum_{i=0}^{j} r_i > j+1$$

- The growth rate of r is the rational number $\frac{p}{q}$.

Definition. Let $r = (r_0, \ldots, r_{q-1})$ be a valid rhythm from this section definition. Then the rhythmic tree $\mathscr{I}_r$ generated by r is defined by:

- the root 0 of $\mathscr{I}_r$ has $(r_0 - 1)$ children, which are the notes $1, 2, \ldots,$ and $(r_0 - 1)$.
- for $n > 0$, the node n has $rn\%q$ children, which are the nodes $(m+1), (m+2), \ldots,$ and $(m + rn\%q)$ where m is the largest child of $(n-1)$.

In [3], they show that because of this rhythmic child count, there are no finite state machines that can describe the language of paths within the k tree. Also, for certain rhythms, this is the same as a rational descending tree. However, each rhythm is associated with a rational number, and so we have a complete k-ary tree like object that works with rational numbers. So, we should find a definition that works for irrational k that isn't simply a k-ary tree.

Also, it is interesting to look at how these trees grow asymptotically. From [3], we see that there isn't a finite state machine that can describe the language of paths in those trees. So, it makes sense to try and figure out the rate at which it is growing. [4] contains a sequence that is the same sequence as the leftmost children in a given k tree. This paper defines the function $c(k)$ which is related to the Josephus problem but also finds relevance with asymptotics in these types of trees.

Preliminaries

Notation

$\mathbb{N}$ denotes the set of non-negative integers, $\mathbb{Q}$ the set of rational numbers, and $\mathbb{R}$ the set of real numbers. $\lfloor \cdot \rfloor$ and $\lceil \cdot \rceil$ denote the floor and ceiling functions. For $x \in \mathbb{R}$, $\{x\} := x - \lfloor x \rfloor$ denotes the fractional part of x. Note that this means that $\{x\} : \mathbb{R} \longrightarrow [0,1)$. $\sim$ denotes asymptotic equivalence, where $f(n) \sim g(n)$ means $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1$.

Definitions

We first formally define a k-descending tree:

Definition. For any $k \in (1,\infty) \subset \mathbb{R}$, the *$k$-descending tree*, or simply *k-tree* is the rooted tree with nodes in $\mathbb{N}$, where every $n \in \mathbb{N}$ has the parent $\lfloor \frac{n}{k} \rfloor$, and 0 is the root node.

Fig. 7, Fig. 8, Fig. 9 were generated with this Figure appendix.

For the integer k case, it generates a complete k-ary tree as shown in Fig. 7.

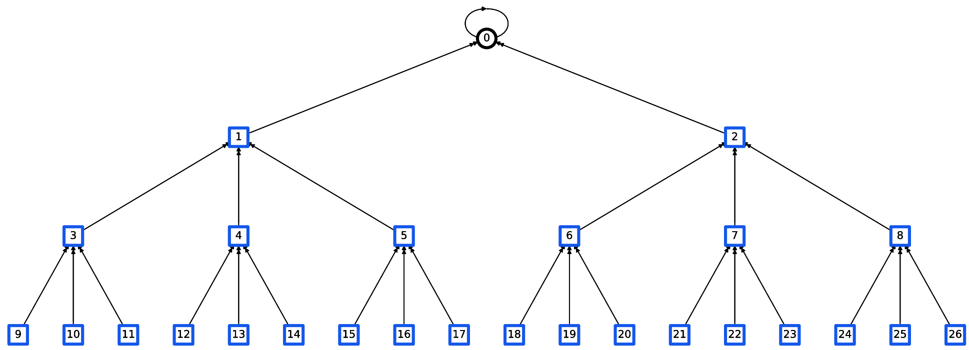

Figure 7: 3-descending tree — Used code to generate this tree.

Then, for rational k, it generates a rhythmic tree as described in [3]. The childcounts in Fig. 8 are periodic with pattern $2,1,\ldots$.

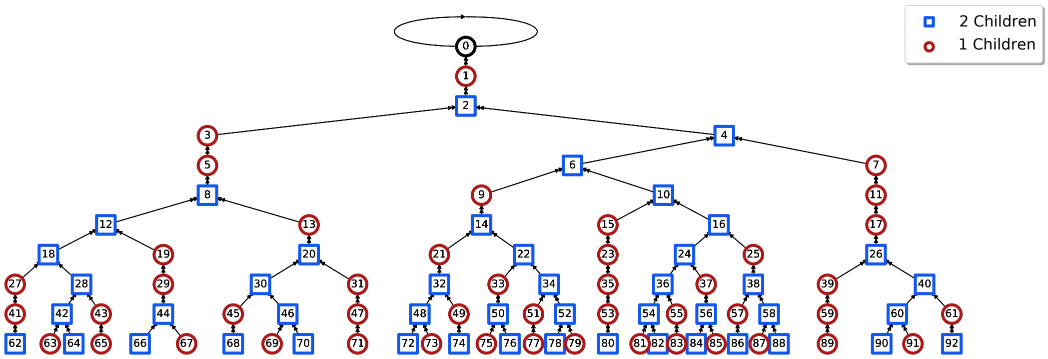

Figure 8: $\frac{3}{2}$-descending tree — Coded it to generate this tree.

Then, when k is irrational, the behavior becomes slightly more chaotic. This specific example in Fig. 9 shows some particularly nice behavior due to our choice of $k = \phi$.

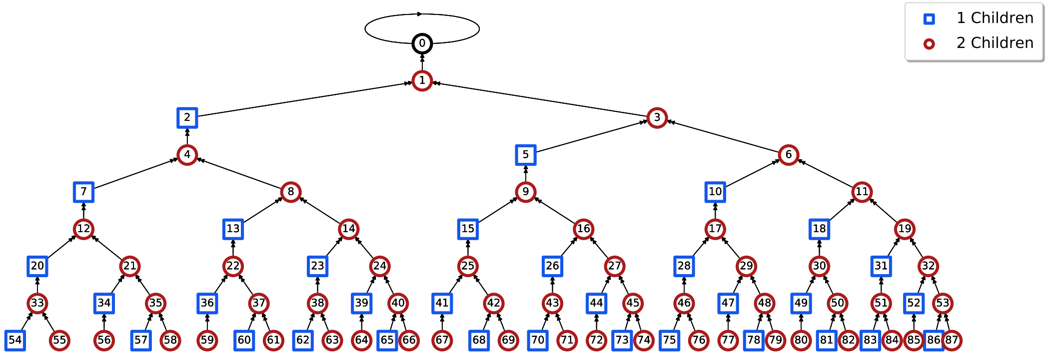

Figure 9: $\frac{1+\sqrt{5}}{2} \approx 1.618$-descending tree — Used a decimal approximation (up to 1000 digits) to generate this tree.

The remaining functions are defined based on a particular k-tree, so are technically also functions of k, but for brevity we will often not write the dependence on k explicitly.

Definition. For any $n \in \mathbb{N}$, let $\mathrm{children}(n) = \{c \in \mathbb{N} : \lfloor \frac{c}{k} \rfloor = n\}$ be the set of children of n in the k-tree[2]. Then $h(n) = |\mathrm{children}(n)|$ gives the *child-count* of n.

A very important first observation is

Remark. For any $n \in \mathbb{N}_+$, $\min(\mathrm{children}(n)) = \lceil n \cdot k \rceil$.

We can now formally prove the claim that the average child-count is k:

Remark. $\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} h(n) = k$.

Proof. Notice that the sum $\sum_{n=0}^{N-1} h(n)$ counts every node from 0 to the largest child of the node $N-1$. Thus,

$$\sum_{n=0}^{N-1} h(n) = 1 + \max(\mathrm{children}(N-1)) = \min(\mathrm{children}(N)) = \lceil N \cdot k \rceil,$$

from **??** . Then $\lim_{N\to\infty} \frac{1}{N} \lceil N \cdot k \rceil = k$. $\square$

The child count $h(n) \in \{\lfloor k \rfloor, \lceil k \rceil\}$, and turns out to depend only on the quantity $\{n \cdot k\}$:

Definition. For $n \in \mathbb{N}_+$, we call the fractional part $\{n \cdot k\} \in [0,1)$ the *count indicator*. The interval $(0, 1-\{k\}] \subset [0,1)$ is called the *floor-range* and its complement $\{0\} \cup (1-\{k\}, 1)$ is called the *ceil-range*.

These definitions are motivated by the following foundational lemma:

[2]Note that technically $0 \in \mathrm{children}(0)$. This makes $h(0) = \lceil k \rceil$, which is consistent with Lemma 1.

Lemma 1. For all $n \in \mathbb{N}$,

$$h(n) = \begin{cases} \lfloor k \rfloor & \textit{if } \{n \cdot k\} \in (0, 1 - \{k\}] \\ \lceil k \rceil & \textit{otherwise.} \end{cases}$$

Proof. This statement is trivial if $k \in \mathbb{Z}$, so we assume now $k \notin \mathbb{Z}$ and $\lfloor k \rfloor + 1 = \lceil k \rceil$.

Notice that $\lfloor \frac{c}{k} \rfloor = n$ iff $c \in [nk, (n+1)k)$, thus $h(n)$ counts the integer points in this interval, whose width is $k = \lfloor k \rfloor + \{k\}$. Intuitively, the first part of the interval $[nk, nk + \lfloor k \rfloor)$ of width $\lfloor k \rfloor$ will always contain $\lfloor k \rfloor$ integer points. Then when $\{nk\} > 1 - \{k\}$, the remaining interval $[nk + \lfloor k \rfloor, nk + k)$ will "wrap around" one additional integer point. On the other hand, when $\{nk\} < 1 - \{k\}$, the fractional part strictly increases and does not wrap around an extra integer point.

In the boundary case $\{nk\} = 0$, $nk \in \mathbb{Z}$, and the interval contains $\lceil k \rceil$ integer points $nk, nk + 1, \ldots, nk + \lfloor k \rfloor$.

In the other boundary case $\{nk\} = 1 - \{k\}$, we have $nk + k \in \mathbb{Z}$, but the interval does not contain this rightmost boundary, and we thus have $\lfloor k \rfloor$ integer points $nk + k - 1, nk + k - 2, \ldots, nk + k - \lfloor k \rfloor$ in the interval. $\square$

Notice that Lemma 1 implies that for rational $k \in \mathbb{Q}$, the child-count function $h(n)$ is periodic. See for example Fig. 8.

Finally, we formally define the row-length sequence:

Definition. $(r_d)_{d=0}^{\infty}$ is the *row-length sequence*, where r_d gives the number of nodes at depth d in a k-tree. More formally, for a node $n \in \mathbb{N}$, the iterated function sequence $g_0 = n$ and $g_{i+1} = \lfloor \frac{g_i}{k} \rfloor$ gives the path to the root. Then $\text{depth}(n) = \max(i : g_i = 0)$ and $r_d = |\{n \in \mathbb{N} : \text{depth}(n) = d\}|$.

The sequence r_d then intuitively grows at an exponential rate of k as how $r_d \approx r_{d-1} \cdot k$. However, due to the rounding off done by the floor function, it is not exactly this. So, we can define an asymptotics function.

Definition. $\rho(k) = \lim_{d\to\infty} \frac{r_d}{k^d}$.

Note that we still need to show this limit exists.

Asymptotics of r_d

Theorem 1. For any $k \in (1, \infty) \subset \mathbb{R}$, the constant $\rho(k) = \lim_{d\to\infty} \frac{r_d}{k^d}$ exists. Thus $r_d \sim \rho(k) \cdot k^d$.

Proof. This is essentially a corollary of Proposition 1 from [4]. To be self-contained, we produce the proof in its entirety.

Observe that for any $n \in \mathbb{N}_+$, $\min(\text{children}(n)) = \lceil n \cdot k \rceil$. We will then consider the sequence $f_0 = 1$ and $f_{i+1} = \lceil f_i \cdot k \rceil$. Notice that, subject to a change in indexing, this gives the leftmost elements in each row. For example, see

f_i	Value
f_0	1
f_1	2
f_2	4
f_3	7
f_4	12
f_5	20

Table 2: f_i values for the ϕ-tree, shown in Fig. 9.

We thus have $r_d = f_d - f_{d-1}$ for all $d > 0$. [4] describes a sequence that is the same sequence as the leftmost children in a given k tree. Proposition 1 shows there exists a constant[3] $c(k)$ such that $f_i \sim c(k) \cdot k^i$. This will then imply

$$r_d \sim c(k) \cdot k^d - c(k) \cdot k^{d-1} = \frac{k-1}{k} c(k) \cdot k^d,$$

thus we have $\rho(k) = \frac{k-1}{k} \cdot c(k)$.

To prove $c(k)$ exists, we consider the sequence $(\frac{f_i}{k^i})_{i=0}^{\infty}$. First we show the sequence is nondecreasing, since

$$\frac{f_{i+1}}{k^{i+1}} = \frac{\lceil f_i \cdot k \rceil}{k^{i+1}} \geq \frac{f_i}{k^i}.$$

The sequence is also bounded above. We start with

$$\frac{f_{i+1}}{k^{i+1}} = \frac{\lceil f_i \cdot k \rceil}{k^{i+1}} < \frac{f_i \cdot k + 1}{k^{i+1}} = \frac{f_i}{k^i} + \frac{1}{k^{i+1}},$$

and then with the base case $\frac{f_0}{k^0} = 1$ we conclude

$$\frac{f_n}{k_n} < \sum_{i=0}^{n} \frac{1}{k^i} < \sum_{i=0}^{\infty} \frac{1}{k^i} = \frac{1}{1 - 1/k} = \frac{k}{k-1}.$$

Since the sequence $(\frac{f_i}{k^i})_{i=0}^{\infty}$ is nondecreasing and bounded from above, the limit $c(k) = \lim_{i \to \infty} (\frac{f_i}{k^i})$ exists. Moreover, we have the following bounds

$$1 \leq c(k) \leq \frac{k}{k-1}.$$

$\square$

Corollary 1. $\frac{k-1}{k} \leq \rho(k) \leq 1$.

Proof. The bounds on $c(k)$ as described in the proof of Theorem 1 alongside the relationship $\rho(k) = \frac{k-1}{k} \cdot c(k)$ gives the following bounds on ρ. $\square$

So, we can now look at approximations of this data. The code for generating these approximations are attached in this appendix. Some of the raw data is contained within this appendix.

[3][4] uses α as the parameter instead of k.

Within [4], Prop 3. they prove that there are jump discontinuities at the "Josephus points," or rational numbers of the form $\frac{q}{q-1}$ where $q \in \mathbb{Z}$. This is shown within Fig. 10.

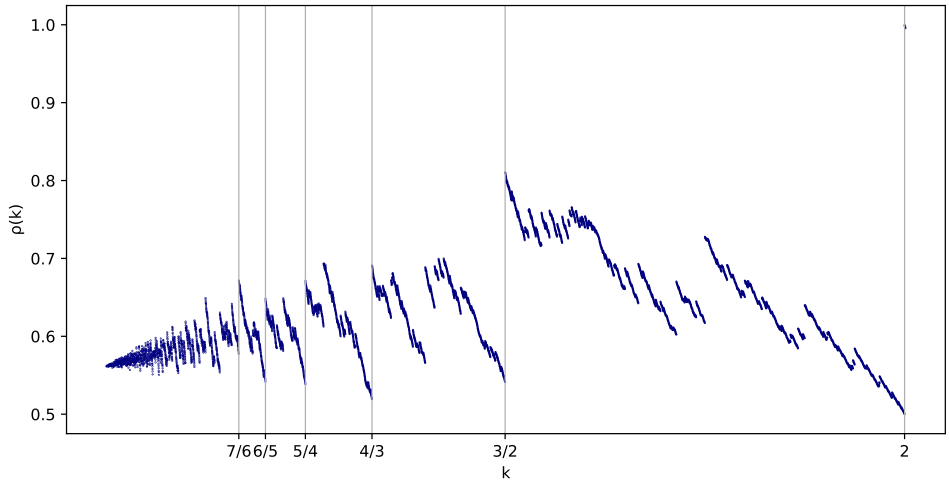

Figure 10: Jump Discontinuities from [4] — In Prop 3. from [4], there are jump discontinuities of an extremely nice form.

Zooming out with Fig. 11, we can see what appears to be a more global pattern.

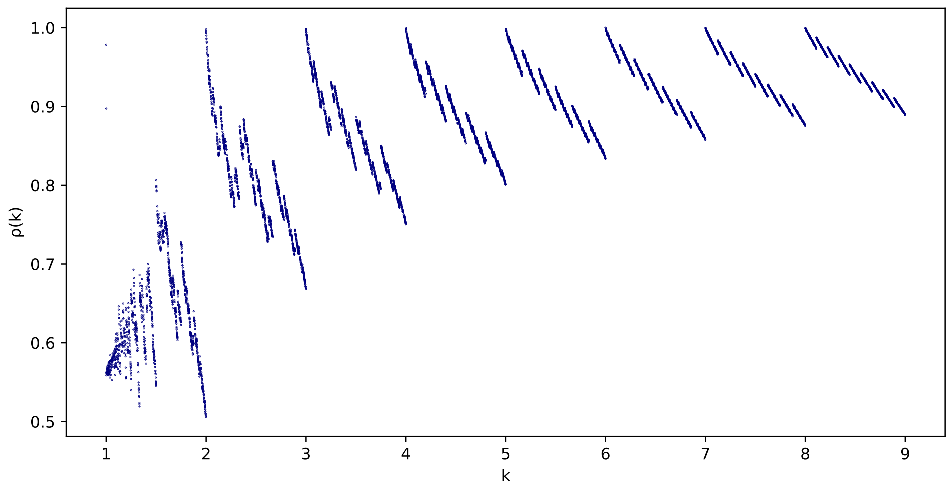

Figure 11: $\rho(k)$ for $1 < k \leq 9$ — Approximated $\rho(k)$ for 10^4 points.

One of these patterns appears to be periodic splits within the function. This is illustrated within Fig. 12. Within n and $(n+1)$ for some integer $n \geq 1$, there are $(n+1)$ periodic 'visible splits.' Upon closer inspection, it appears there are also $(n+1)^2$ periodic 'visible splits,' with the best ones happening closest to $(n+1)$. This pattern visually holds, and the splits get smaller.

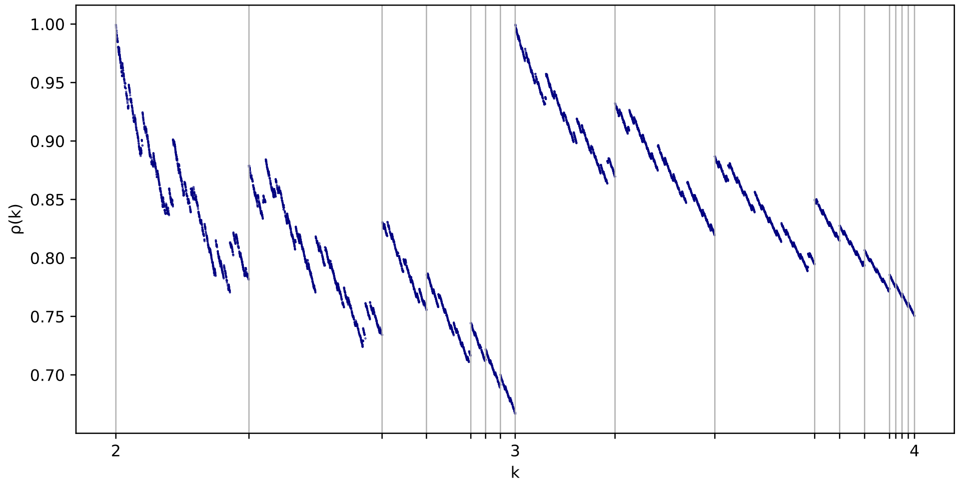

Figure 12: $\rho(k)$ for $1 < k \leq 4$ with apparent splits — Approximated $\rho(k)$ for 10^4 points, and also plotted the points of possible visible splits that seem to be self similar between n and $n+1$.

k-Trees with Closed Form $\rho(k)$

Lemma 2. Let k satisfy the equation $k = a + \frac{b}{k}$ for $a, b \in \mathbb{Z}$. Then for any node $n \in \mathbb{N}_+$, let $x = \{n \cdot k\} \in [0, 1)$ be the count-indicator for n. Let $c_1, \ldots, c_{h(n)}$ be the children of the node n. Then the ith smallest child c_i has count-indicator

$$\{c_i \cdot k\} = \{(i - x) \cdot \frac{b}{k}\}.$$

Proof. From **??** , we have $c_1 = \lceil n \cdot k \rceil$, and more generally $c_i = \lceil n \cdot k \rceil + i - 1$. We then have count indicator

$$\begin{aligned}
\{c_i \cdot k\} &= \{(\lceil n \cdot k \rceil + i - 1)(a + \frac{b}{k})\} \\
&= \{(\lceil n \cdot k \rceil + i - 1) \cdot \frac{b}{k})\} \\
&= \{(n \cdot k + 1 - \{n \cdot k\} + i - 1) \cdot \frac{b}{k})\} \\
&= \{(i - x) \cdot \frac{b}{k}\}.
\end{aligned}$$

These are shown in Fig. 16, Fig. 17, Fig. 18, Fig. 19, Fig. 20, located in this appendix. □

Theorem 2. This is the grandparent theorem. Let $a, b \in Z$ such that $a \geq 1$ and $1 - a \leq b \leq a - 1$. When $b \geq 0$, then there are always b distinct lines in the ceil-range of the child count indicator graph for $k = \frac{a + \sqrt{a^2 + 4b}}{2}$. When b is negative, then there are always $|b|$ distinct lines in the floor-range of the graph of k.

Proof. Let $a, b \in Z$ such that $a \geq 1$ and $1-a \leq b \leq a-1$. Then, let $k = \frac{a+\sqrt{a^2+4b}}{2}$. Using Lemma 2,

$$\begin{aligned}
\{c_{i+1} \cdot k\} - \{c_i \cdot k\} &= \{(i+1-x) \cdot \frac{b}{k}\} - \{(i-x) \cdot \frac{b}{k}\} \\
&= \{\frac{b}{k}\} \\
&= \{k\}, 1 - \{k\} (\text{depending on } b).
\end{aligned}$$

So, the difference between two consecutive childcount indicators is $\{k\}$ or $\{1-k\}$. The output ceil-range from Lemma 1 is of size $\{k\}$, and the floor-range is then $1 - \{k\}$. So, when one childcount indicator falls above or under that line, the next indicator "flips," meaning that the amount of lines in the output ceil-range and floor-range stays constant throughout $x \in [0, 1)$. So, this simplifies the proof in allowing us to choose for x to show that there are specifically b indicators in the specific output range.

The last step is now to find the number of solutions for i in this expression,

$$1 - \{k\} < \{\{k\}(i-x)\} < 1,$$

but now we can choose x. If $x = 0$, then the set of solutions looks like

$$\{\lfloor \frac{a}{b} \rfloor, \lfloor \frac{2a}{b} \rfloor, \lfloor \frac{3a}{b} \rfloor, \ldots, \lfloor \frac{(b-1)a}{b} \rfloor, \lfloor a \rfloor\}$$

, which contains b solutions (in the nonnegative b case). This is the set of solutions to $0 < \{\{k\}(i - x)\} < 1 - \{k\}$ for the negative b case. This then proves the theorem. □

Theorem 3. Let $a, b \in \mathbb{Z}$ with $a \geq 1$ and $1-a < b < 1+a$. For $k = \frac{a+\sqrt{a^2+4b}}{2}$, the row-length sequence for the k-tree satisfies the linear recurrence

$$r_d = a \cdot r_{d-1} + b \cdot r_{d-2},$$

with base case $r_0 = 1$, $r_1 = \lceil k \rceil - 1$.

Proof. Let $a, b \in \mathbb{Z}$ with $a \geq 1$ and $1-a < b < 1+a$. When $b \geq 0$, we can see that $\lfloor k \rfloor = a$. So, $r_d \geq a r_{d-1}$, as from Lemma 1 we know that each element from the previous row contributes at least $\lfloor k \rfloor$ children. With Theorem 2, we can see that each element in r_{d-2} has b children that have $\lceil k \rceil$ children, which creates the equality $r_d = a r_{d-1} + b r_{d-2}$. Instead, when $b < 0$, then $\lceil k \rceil = a$, so $r_d \leq a r_{d-1}$. What happens then is $b r_{d-2}$ actually takes away to compensate for the values with $\lfloor k \rfloor$ children. So,

$$r_d = a r_{d-1} + b r_{d-2}.$$

□

Corollary 2. Let $a, b \in \mathbb{Z}$ with $a \geq 1$ and $1-a < b < 1+a$. For $k = \frac{a+\sqrt{a^2+4b}}{2}$,

$$\rho(k) = \begin{cases} \dfrac{k}{\sqrt{a^2+4b}} & b > 0 \\[2ex] \dfrac{k-1}{\sqrt{a^2+4b}} & b < 0 \\[2ex] \dfrac{a-1}{a} & b = 0 \end{cases}$$

Proof. Because of Theorem 3, we can find a closed formula for r_d.

As the theorem states, let $a,b \in \mathbb{Z}$ such that $a \geq 1, 1-a < b < 1+a$, and let $k_1 = \frac{a+\sqrt{a^4+4b}}{2}$ and $k_2 = \frac{a-\sqrt{a^4+4b}}{2}$.

Then, for the k_1 tree,

$$r_d = ar_{d-1} + br_{d-2}.$$

Like any recurrence problem, let $r_d = c^d$ for some constant c. Then,

$$\begin{aligned} c^d &= ac^{d-1} + bc^{d-2} \\ c^2 &= ac + b \\ c^2 - ac - b &= 0 \\ c &= k_1, k_2. \end{aligned}$$

So, we can plug in c to r_d.

$$r_d = Ak_1^d + Bk_2^d,$$

for some constants A and B.

Now, we have to use the starting conditions $r_0 = 1, r_1 = \lfloor k \rfloor$ to find A and B. This is simply done by plugging in $d = 0, 1$. However, note that when $b \geq 0$, $\lfloor k \rfloor = a$, and when $b < 0$, then $\lfloor k \rfloor = a - 1$.

So, when $b \geq 0$, we can solve a matrix.

$$\begin{bmatrix} 1 & 1 & 1 \\ k_1 & k_2 & a \end{bmatrix} \to \begin{bmatrix} 1 & 0 & \dfrac{a+\sqrt{a^2+4b}}{2\sqrt{a^2+4b}} = \dfrac{k_1}{\sqrt{a^2+4b}} \\ \\ 0 & 1 & \dfrac{-a+\sqrt{a^2+4b}}{2\sqrt{a^2+4b}} = \dfrac{-k_2}{\sqrt{a^2+4b}} \end{bmatrix}$$

So, if $b \geq 0$,

$$r_d = \frac{k_1^{d+1} - k_2^{d+1}}{\sqrt{a^2+4b}}.$$

By solving the same original matrix with $a-1$ swapped with a, we can get that when $b < 0$,

$$r_d = \frac{(k_1-1)^{d+1} - (k_2-1)^{d+1}}{\sqrt{a^2+4b}}.$$

Now, we can plug these closed formulas into $\rho(k)$. When $b \geq 0$,

$$\begin{aligned} \rho(k) &= \lim_{d\to\infty} \frac{r_d}{k_1^d} \\ &= \lim_{d\to\infty} \frac{k_1^{d+1} - k_2^{d+1}}{k_1^d \sqrt{a^2+4b}} \\ &= \lim_{d\to\infty} \frac{k_1^{d+1}}{k_1^d \sqrt{a^2+4b}} \\ &= \frac{k_1}{\sqrt{a^2+4b}}. \end{aligned}$$

Similarly, when $b < 0$,

$$\rho(k) = \frac{k_1 - 1}{\sqrt{a^2 + 4b}}.$$

Note that when $b = 0$, then $k = a \in \mathbb{Z}$, so we can also write $\rho(k) = \dfrac{a-1}{a}$ due to $r_d = k^{d-1}(k-1)$. □

The values described in Corollary 2 are shown in Table 3.

	a = -1	a = 0	a = 1	a = 2	a = 3	a = 4	a = 5	a = 6	a = 7
b = 8	2.3723	2.8284	3.3723	4	4.7016	5.4641	6.2749	7.1231	8
b = 7	2.1926	2.6458	3.1926	3.8284	4.5414	5.3166	6.1401	7	7.8875
b = 6	2	2.4495	3	3.6458	4.3723	5.1623	6	6.873	7.772
b = 5	1.7913	2.2361	2.7913	3.4495	4.1926	5	5.8541	6.7417	7.6533
b = 4	1.5616	2	2.5616	3.2361	4	4.8284	5.7016	6.6056	7.5311
b = 3	1.3028	1.7321	2.3028	3	3.7913	4.6458	5.5414	6.4641	7.4051
b = 2	1	1.4142	2	2.7321	3.5616	4.4495	5.3723	6.3166	7.2749
b = 1	0.618	1	1.618	2.4142	3.3028	4.2361	5.1926	6.1623	7.1401
b = 0	0	0	1	2	3	4	5	6	7
b = -1				1	2.618	3.7321	4.7913	5.8284	6.8541
b = -2					2	3.4142	4.5616	5.6458	6.7016
b = -3						3	4.3028	5.4495	6.5414
b = -4						2	4	5.2361	6.3723
b = -5							3.618	5	6.1926
b = -6							3	4.7321	6

Table 3: Values of the function $k = \frac{a+\sqrt{a^2+4b}}{2}$ for $a, b \in \mathbb{Z}$. Calculated the formula $k = \dfrac{a+\sqrt{a^2+4b}}{2}$ with decimal approximations for $-1 \le a \le 7$, and $-6 \le b \le 8$.

So, by using Corollary 2, we can look at the closed formula for points that we do know on Fig. 13.

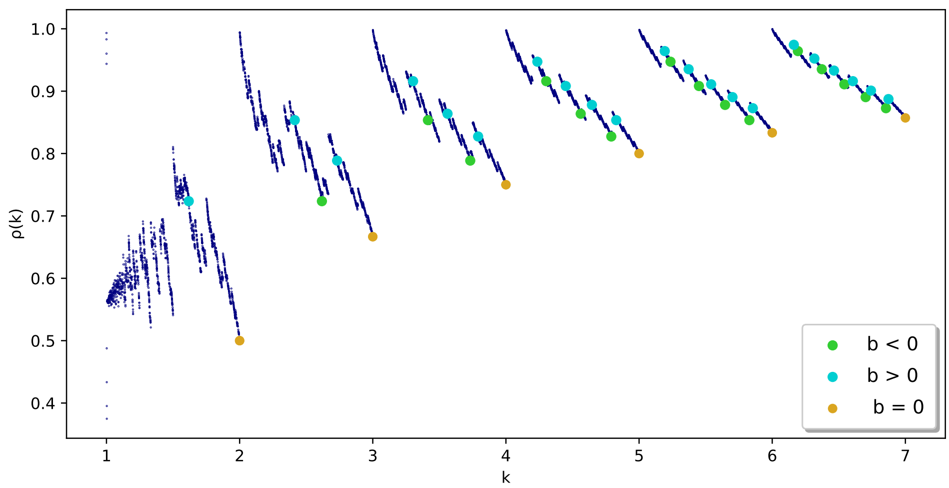

Figure 13: $\rho(k)$ for $1 < k \leq 7$ with closed form — Approximated $\rho(k)$ for 10^4 points, and also plotted the points that contain a closed formula.

Conclusion

So, we have found a closed formula for the irrational case, even when the rational case doesn't have the same type of behavior. This is a rare case where irrationality seems to behave more nicely than their rational counterpart. Also, we have a closed formula for these golden-like trees. This is interesting and motivates the extension from a k-ary tree.

Now, for future work, there seems to be three different modes of progress. First would simply be to change from $\lfloor \frac{n}{k} \rfloor$ to $\lceil \frac{n}{k} \rceil$. This would be a k-ascending tree.

Definition. For any $k \in (1, \infty) \subset \mathbb{R}$, the *k-ascending tree* is the rooted tree with nodes in $\mathbb{N}$, where every $n \in \mathbb{N}$ has the parent $\lceil \frac{n}{k} \rceil$.

Note that the root node of this Figure definition is not going to be 0.

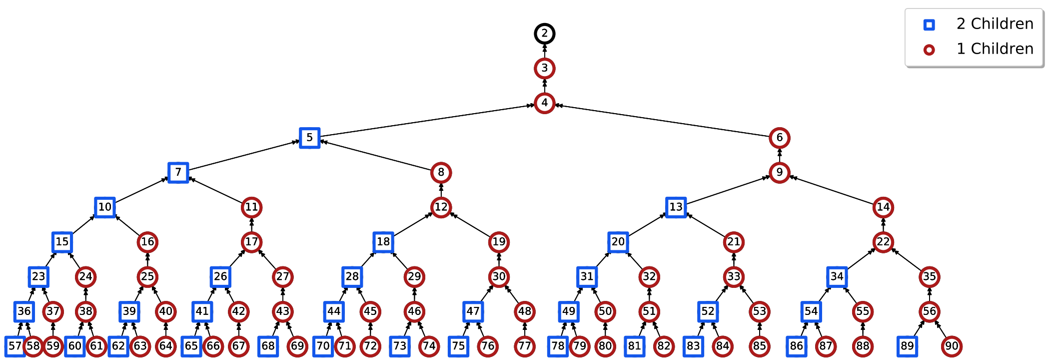

Figure 14: ϕ-ascending tree — This generates a ϕ-ascending tree. Modified the original code to use the ceiling function instead of floor

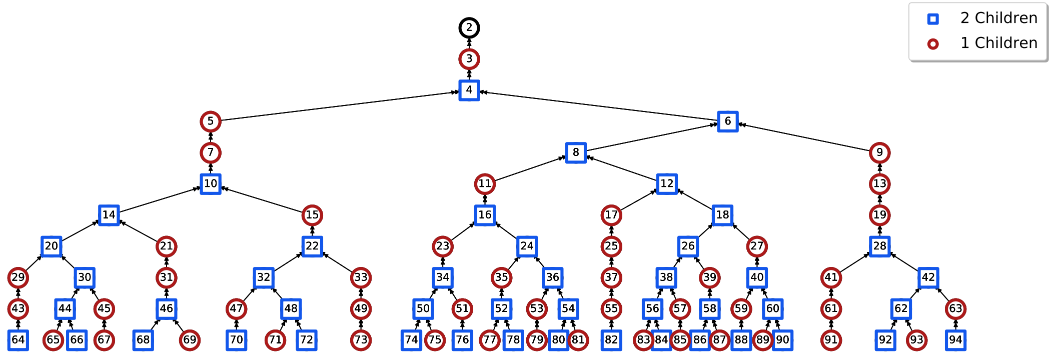

Figure 15: $\frac{3}{2}$-ascending tree — This generates a $\frac{3}{2}$-ascending tree. Modified the original code to use the ceiling function instead of floor

From observation, both examples seem to be shifted (as the root is not 0), and Fig. 14 also seems to be flipped. From this it seems that a lot of the same ideas would carry over very nicely.

Second, it may be interesting to look at more complex k's of the form $k = a + \frac{b}{k} + \frac{c}{k^2}$. Instead of Theorem 3's $r_d = a \cdot r_{d-1} + b \cdot r_{d-2}$, we may have the recurrence $r_d = a \cdot r_{d-1} + b \cdot r_{d-2} + c \cdot r_{d-3}$. It would be nice if this was true, but it doesn't seem like it would work due to the theorem's reliance on the grandparent indicator only going back to r_{d-2}.

Finally, because of the lack of finite state machines for most irrational cases and most rational cases [3], it would be interesting to see how a fractional base could work. [1] talks about fractional bases, using an exploding dots metaphor. The path to a node n in the k-tree could be seen as some sort of base-k representation of n, but addition would have to be redefined as it can generate incorrect paths in the tree.

Within this paper there was the surprising connection to the Josephus problem [4], and the article [2] looks specifically at the constant $\rho(\frac{3}{2})$ in regard to the Collatz Conjecture and mentions the Josephus problem. This connection can be further fleshed out if one found a more direction connection by figuring out how to represent the Josephus problem with trees.

Bibliography

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[2] James Grime, Kevin Knudson, Pamela Pierce, Ellen Veomett, and Glen Whitney. Beyond pi and e: a collection of constants. *Math Horizons*, 29(1):8–12, 2022.

[3] Victor Marsault and Jacques Sakarovitch. Rhythmic generation of infinite trees and languages. *arXiv preprint arXiv:1403.5190*, 2014.

[4] Andrew M Odlyzko and Herbert S Wilf. Functional iteration and the josephus problem. *Glasgow Mathematical Journal*, 33(2):235–240, 1991.

Tree Generation Code

This is the code used to generate the *k*-tree figures.

```
from decimal import *
from re import X
import networkx as nx
import matplotlib.pyplot as plt
from math import floor, ceil
import os
import random

save_as = 4
save_location = "./drive/MyDrive/Figures"
largest_can_draw = 1000

os.makedirs(save_location, exist_ok=True)

def rows(n, k):
    """ returns r0...r_n in the k-tree """
    r = row_gen(Decimal(k))
    return [next(r) for x in range(n)]

def left_gen(k):
  """ generator for leftmost nodes in the k-tree """
  yield 0
  l = 1
  while True:
    yield l
    l = ceil(Decimal(l) * Decimal(k))

def row_gen(k):
  """ generator for row lengths in the k-tree"""
  left_iter = left_gen(k)
  last_left = next(left_iter)
  while True:
    next_left = next(left_iter)
    yield next_left - last_left
    last_left = next_left

def fancytree(largest, k, save = True):
    """
    Draws a k-tree up to largest value given.
    To use depth instead, pass on the max_n from here:

    depth = 10 # some number of rows
    max_n = 0
    row_lengths = rows(depth, k)
    for row in row_lengths:
        max_n += row

    """
    G = nx.DiGraph()
    fl = []
    ce = []

```

```
    # CONNECTING THE NODES
    for x in range(1, largest):
        G.add_edge(floor(x/k), x)
        if Decimal(k*x) - floor(Decimal(k*x)) < 1 - k + floor(k):
            fl.append(x)
        else:
            ce.append(x)

    # DRAWING THE TREE
    if (largest > largest_can_draw):
        return
    plt.clf()
    pos = hierarchy_pos(nx.Graph(G), 0)
    G.add_edge(0, 0)

    # DRAWS THE NODES WITH COLOR
    nsize = 450
    lwidth = 3.5
    nx.draw(G.reverse(), pos, nodelist = [0], node_size = nsize, node_color = '#ffffff', node_shape = "o", edgecolors='#000000', linewidths = lwidth, with_labels = True)
    nx.draw(G.reverse(), pos, label = str(floor(k)) + " Children", nodelist = fl, node_size = nsize, node_color = '#ffffff', node_shape = "s", edgecolors='#1357eb', linewidths = lwidth, with_labels = True)
    nx.draw(G.reverse(), pos, label = str(ceil(k)) + " Children", nodelist = ce, node_size = nsize, node_color = '#ffffff', node_shape = "o", edgecolors='#ab1a1a', linewidths = lwidth, with_labels = True)
    if abs(k - int(k)) <= 10**(-5):
      pass
    else:
      plt.legend(loc = 'best', fontsize = 'xx-large', markerscale = .45,
                 labelspacing = .7,
                 title_fontsize = 'xx-large', borderpad = .6, shadow = True)
    if save:
      plt.savefig(save_location+str(round(k, save_as))+"tree.pdf", format = 'pdf', dpi=300)
    plt.show()

def hierarchy_pos(G, root=None, width=1., vert_gap = 0.2, vert_loc = 0, leaf_vs_root_factor = 0.5):
    """
    Source: https://github.com/ryoji-kubo/CurvatureAnalysis/blob/main/utils.py
    Source: https://github.com/BaseMax/BinaryTreeDiagramDrawing/blob/master/tree.py
    """

    if root is None:
        if isinstance(G, nx.DiGraph):
            root = next(iter(nx.topological_sort(G)))  #allows back compatibility with nx version 1.11
        else:
```

```
            root = random.choice(list(G.nodes))

    if not nx.is_tree(G):
        raise TypeError('cannot use hierarchy_pos on a graph that is not a tree')

    def _hierarchy_pos(G, root, leftmost, width, leafdx = 0.2, vert_gap = 0.2, vert_loc = 0,
                    xcenter = 0.5, rootpos = None,
                    leafpos = None, parent = None):

        if rootpos is None:
            rootpos = {root:(xcenter,vert_loc)}
        else:
            rootpos[root] = (xcenter, vert_loc)
        if leafpos is None:
            leafpos = {}
        children = list(G.neighbors(root))
        leaf_count = 0
        if not isinstance(G, nx.DiGraph) and parent is not None:
            children.remove(parent)
        if len(children)!=0:
            rootdx = width/len(children)
            nextx = xcenter - width/2 - rootdx/2
            for child in children:
                nextx += rootdx
                rootpos, leafpos, newleaves = _hierarchy_pos(G,child, leftmost+leaf_count*leafdx,
                                    width=rootdx, leafdx=leafdx,
                                    vert_gap = vert_gap, vert_loc = vert_loc-vert_gap,
                                    xcenter=nextx, rootpos=rootpos, leafpos=leafpos, parent = root)
                leaf_count += newleaves

            leftmostchild = min((x for x,y in [leafpos[child] for child in children]))
            rightmostchild = max((x for x,y in [leafpos[child] for child in children]))
            leafpos[root] = ((leftmostchild+rightmostchild)/2, vert_loc)
        else:
            leaf_count = 1
            leafpos[root]  = (leftmost, vert_loc)
        return rootpos, leafpos, leaf_count

    xcenter = width/2.
    if isinstance(G, nx.DiGraph):
        leafcount = len([node for node in nx.descendants(G, root) if G.out_degree(node)==0])
    elif isinstance(G, nx.Graph):
        leafcount = len([node for node in nx.node_connected_component(G, root) if G.degree(node)==1 and node != root])
    rootpos, leafpos, leaf_count = _hierarchy_pos(G, root, 0, width,
                                                    leafdx=width*1./leafcount,
```

```
                                                    vert_gap=vert_gap,
                                                    vert_loc = vert_loc,
                                                    xcenter = xcenter)
    pos = {}
    for node in rootpos:
        pos[node] = (leaf_vs_root_factor*leafpos[node][0] + (1-leaf_vs_root_factor)*rootpos[node][0], leafpos[node][1])
    xmax = max(x for x,y in pos.values())
    for node in pos:
        pos[node]= (pos[node][0]*width/xmax, pos[node][1])
    return pos
```

Listing 1: Tree Generation Code

ρ Code

This is the code used to generate $\rho(k)$ values.

```
import pandas as pd
import numpy as np
from math import floor, ceil, sqrt, log
from decimal import *
import matplotlib.pyplot as plt

source = "./drive/MyDrive/Figures/rho.csv"

def resetRho():
  """ Empties out if error is encountered """
  df = pd.DataFrame({'val':[],'out':[],'precision':[]})
  df.to_csv(source, index=False)
  return df

def getPrec(k, df):
  """ Returns the stored precision for rho(k) in the csv. 0 if not found """
  try:
    return df.loc[min(df.loc[df.val == k].index)].precision
  except:
    return 0

def addRho(df, info):
  """ Adds value to the stored csv. """
  inside = getPrec(info[0], df) != 0
  if inside:
    df.loc[(df["val"] == info[0])] = info
  else:
    df.loc[len(df)] = info
  return df

def writeRho(df):
  """ Writes to an external csv for storage """
  df.to_csv(source, index = False)

def left_gen(k):
```

```
  """ Generator for leftmost nodes """
  yield 0
  l = 1
  while True:
    yield l
    l = ceil(Decimal(l) * Decimal(k))

def row_gen(k):
  """ Generator for row lengths """
  left_iter = left_gen(k)
  last_left = next(left_iter)
  while True:
    next_left = next(left_iter)
    yield next_left - last_left
    last_left = next_left

def get_rho(k, min_row_size = 10**3, eps = 10 ** -3, iter_upper_bound = 10 ** 2):
  """
  If a rho(k) value is not found within rho.csv, generate it.
  Also runs if a higher precision value
  precision is defined with:
  minimum row size to start at, upper bound to stop at, eps, decimal precision
  """
  if k <= 1:
    return None
  elif abs(k - int(k)) < 10**(-1000000):
    return Decimal((k-1))/Decimal(k)
  row = row_gen(k)
  last_c = 1
  r = next(row)
  c = 1
  while r < min_row_size:
    r = next(row)
    c += 1
  for i in range(c, iter_upper_bound+c):
    next_c = Decimal(next(row)) / Decimal(k) ** Decimal(i)
    if abs(next_c - last_c) < eps:
      return (next_c + last_c) / 2
    last_c = next_c
  return next_c

def find_rho(k, min_row_size = 10**3, eps = 10 ** -3, iter_upper_bound = 10 ** 2):
  """
  Gets the value of rho(k) to some precision, either through rho.csv or generating it.
  """
  try:
    df = pd.read_csv(source)
  except FileNotFoundError:
    return get_rho(k, min_row_size=min_row_size, eps=eps, iter_upper_bound=iter_upper_bound)
  except:
    df = resetRho()
```

```
  if abs(k - int(k)) < 10**(-100):
    prec = np.inf
  else:
    prec = log(min_row_size*eps**(-1)*iter_upper_bound*getcontext().prec, 10)
  curPrec = getPrec(k, df)
  if curPrec < prec or curPrec == 0:
    rho = get_rho(k, min_row_size=min_row_size, eps=eps, iter_upper_bound=iter_upper_bound)
    writeRho(addRho(df, [k, rho, prec]))
    return rho
  elif curPrec >= prec:
    return df.loc[min(df.loc[df.val == k].index)].out

x = np.linspace(1 + 10**(-6), 6, 10**5)
plt.scatter(x, np.array(list(map(find_rho, x))), s = 0.1)
plt.xlabel('k')
plt.ylabel('  (k)')
plt.title(f'Estimated values of   (k) for 100000 random values 1 < k < 6')
plt.show()
```

Listing 2: $\rho(k)$ Approx. Code

ρ Raw Data

This is some recorded raw data generated by the code. Note that there appears to be some rounding off that happens as the data is written to the csv file. The data is actually too large to be typically viewed with tools like csvsimple. The link is at https://github.com/agniv-the-marker/rho-stuff.

Grandparent Indicator Figures

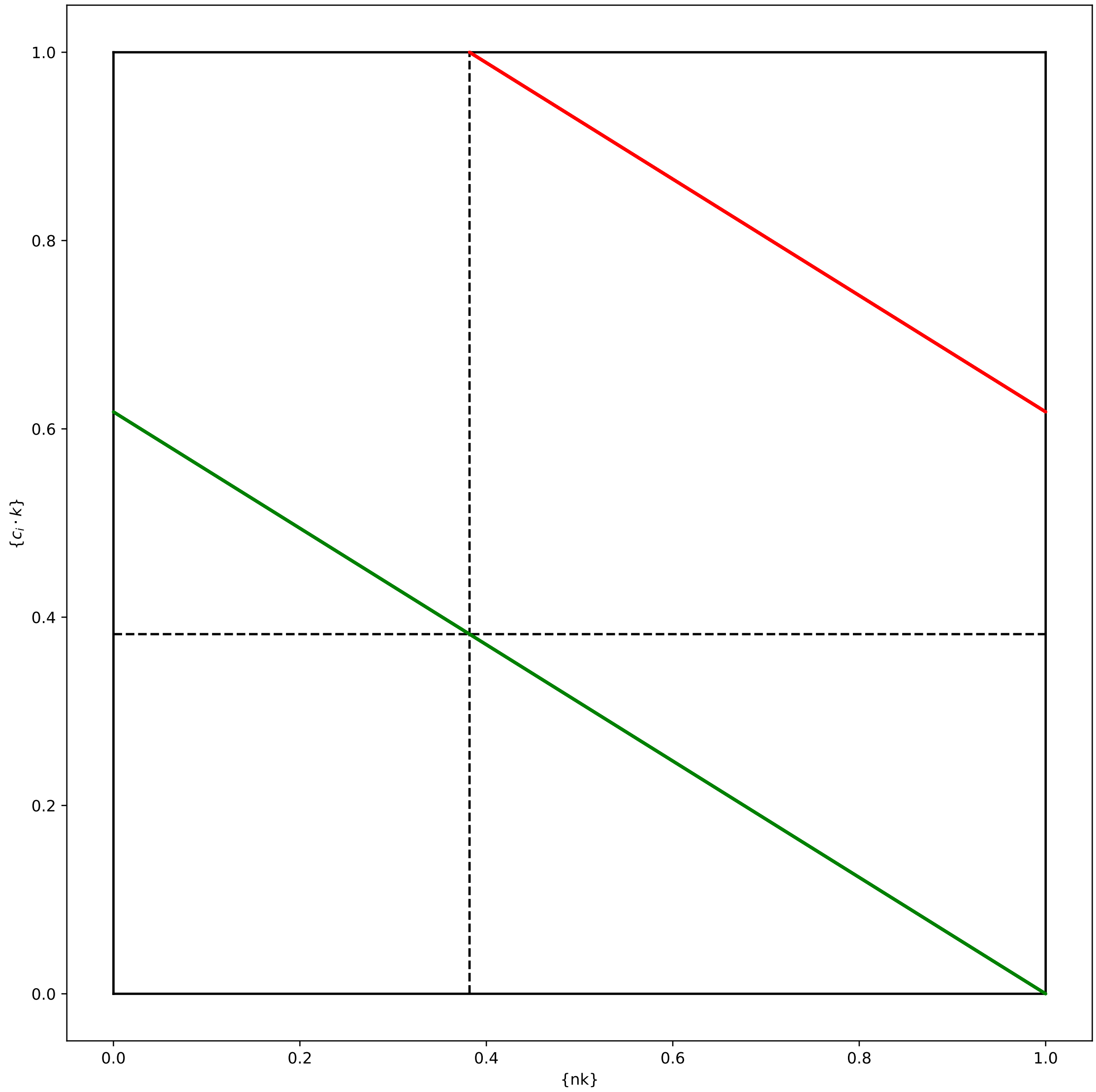

Figure 16: ϕ Grandparent Indicators

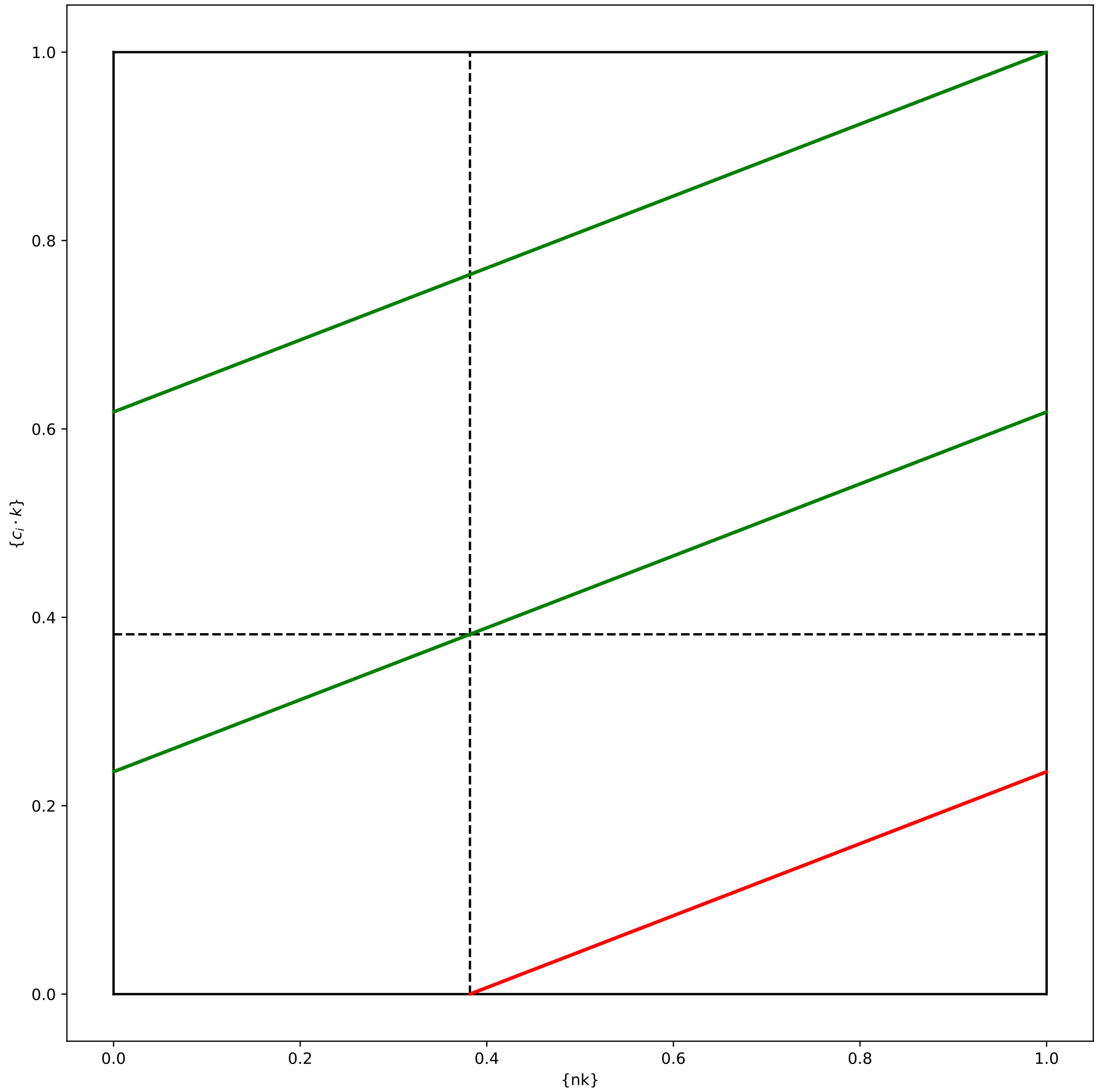

Figure 17: $a = 3, b = -1$ Grandparent Indicators

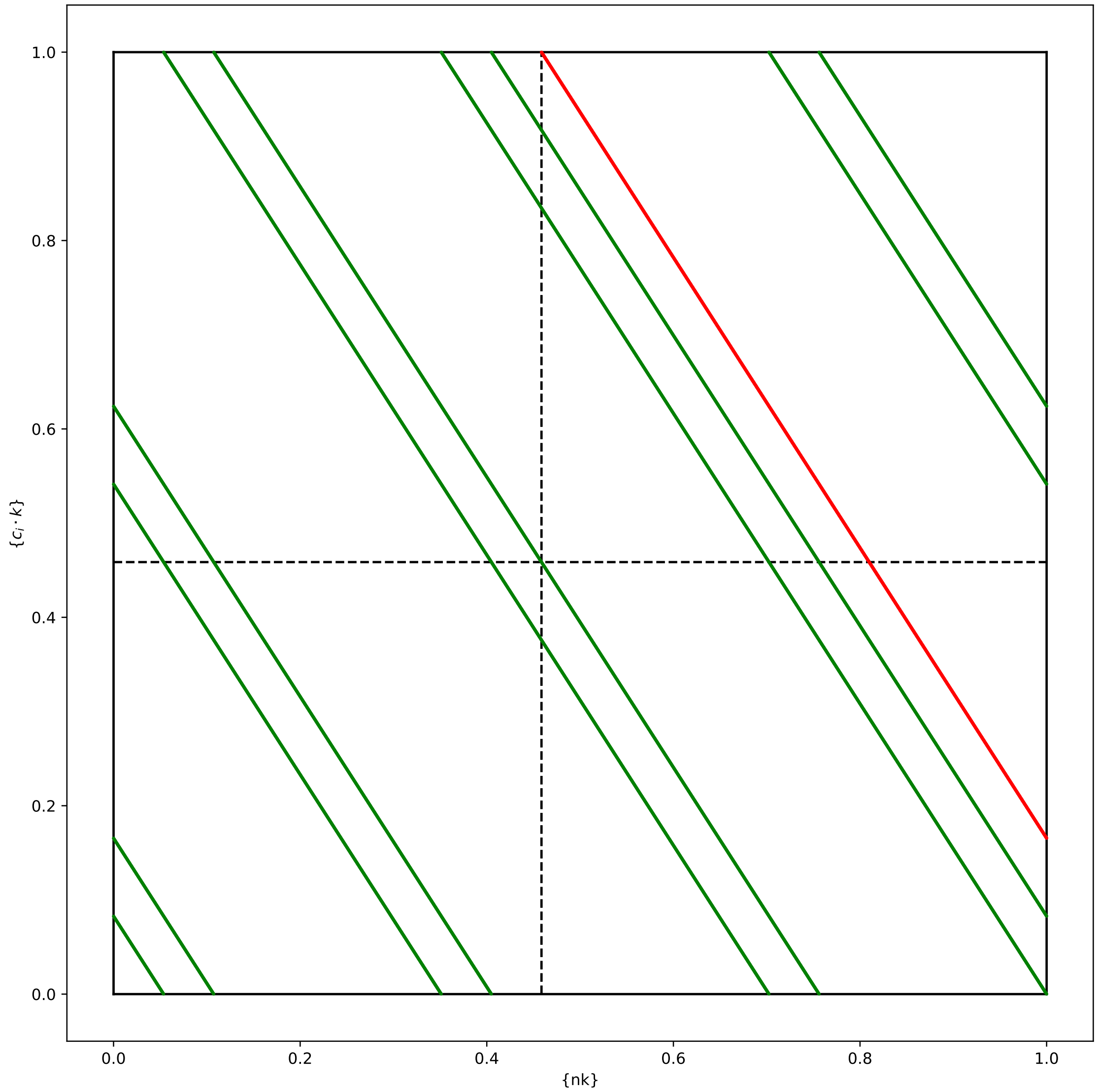

Figure 18: $a = 3, b = 7$ Grandparent Indicators

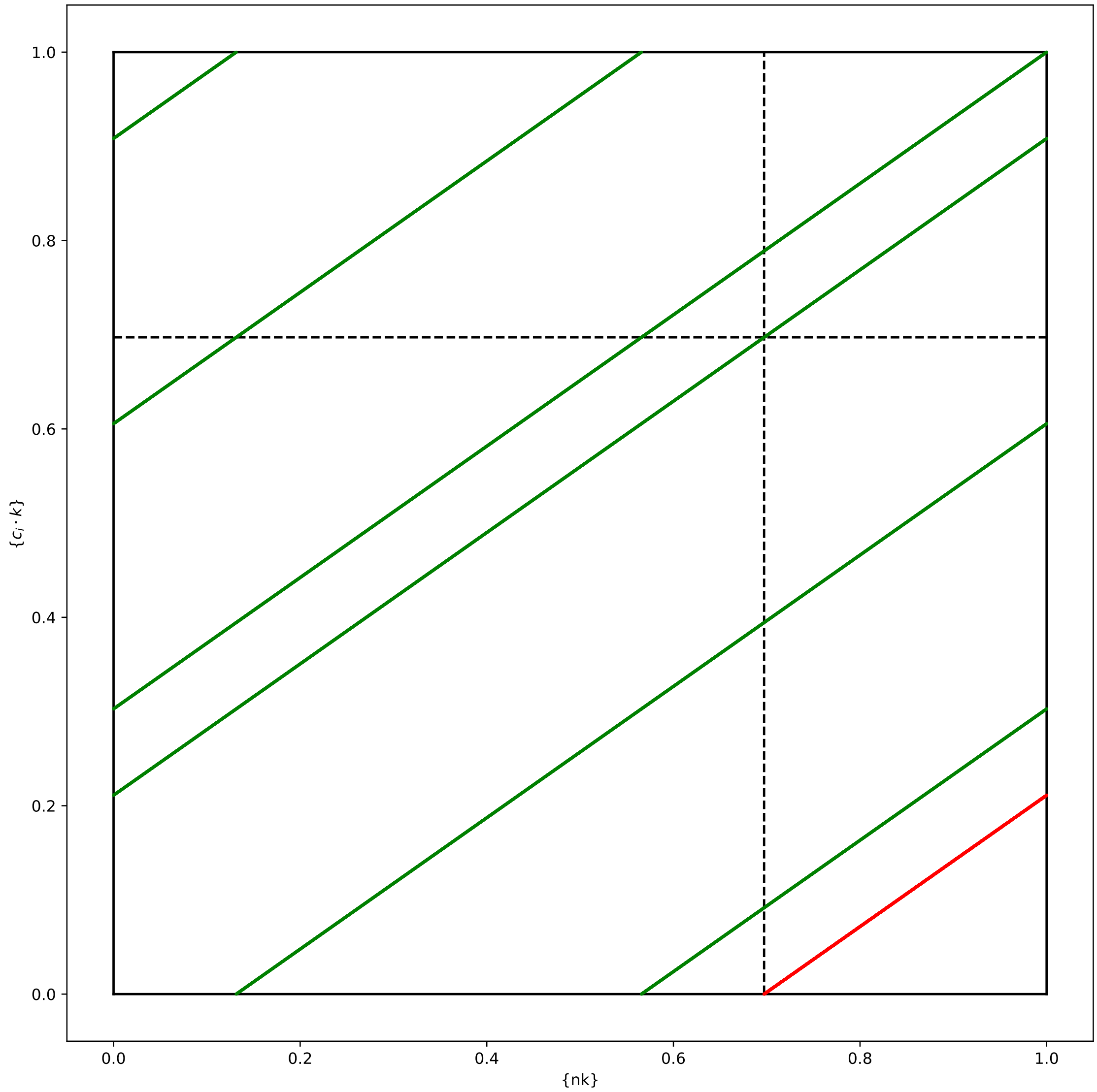

Figure 19: $a = 5, b = -3$ Grandparent Indicators

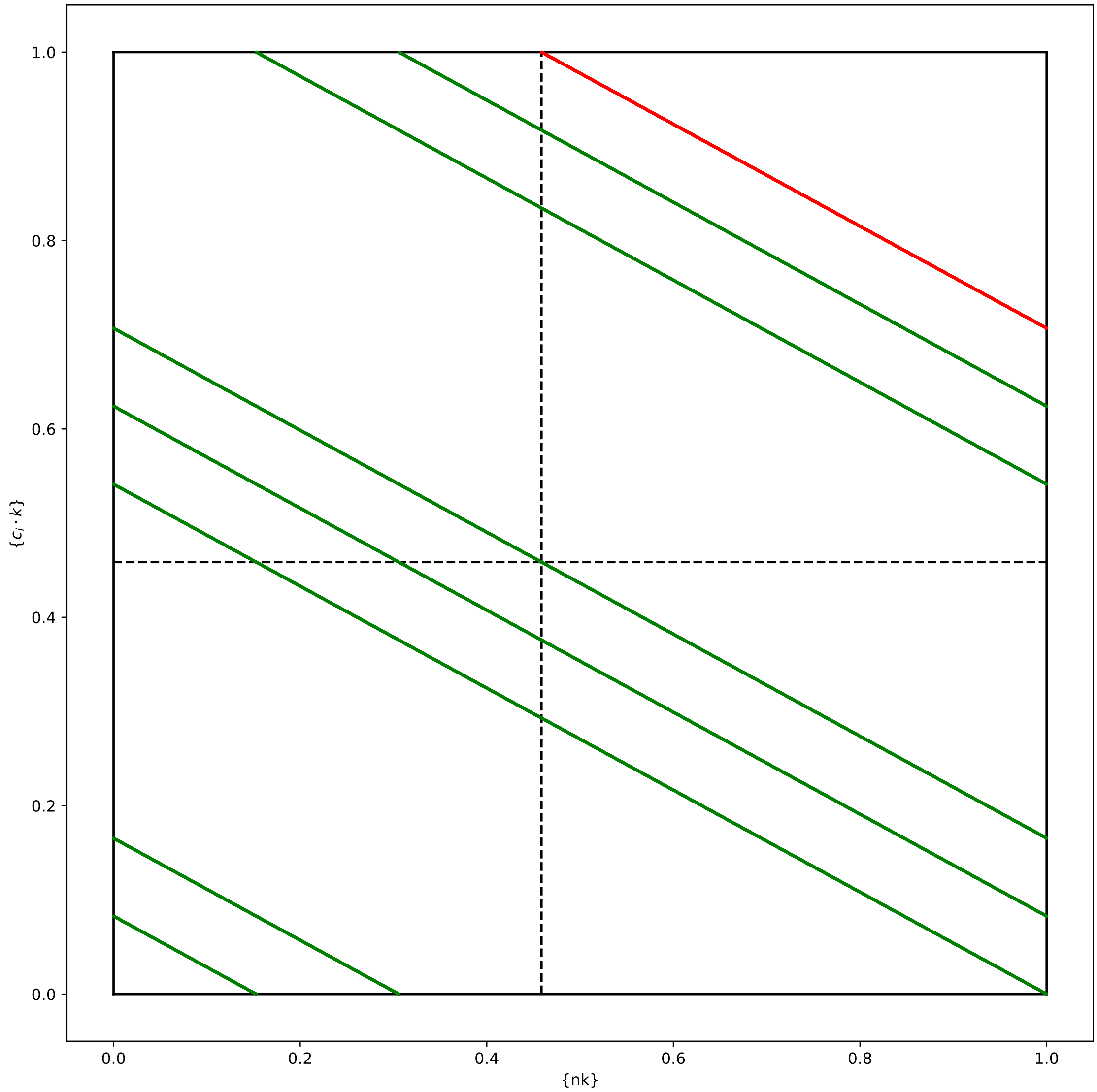

Figure 20: $a = 5, b = 3$ Grandparent Indicators

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Puzzles in Coin Weighing

Dick Hess and Andy Liu

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Abstract
In a collection of coins there may be some that are different. We use a two-pan balance to study when they can be identified.

We first establish some terminology. We do not use the adjectives real, fake, genuine or counterfeit as they all have irrelevant connotations. Instead, we have normal coins all of which have the same weight. There may be abnormal coins each heavier than a normal coin. There may also be abnormal coins each lighter than a normal coin. Any difference in weight is insignificant compared to the weight of a normal coin. There is no need to say that the coins have identical appearance. If each has a unique appearance instead, the problem and the solution will not be affected.

We now establish some notation. We use capital letters to label the coins. We use a two-pan balance which only indicates which pan, if either, contains coins with a higher total weight. We always put the same number of coins in each pan. Coins in the same pan are enclosed in brackets. The brackets are omitted if the number of coins in the pan is one. When (A,B) is compared with (C,D), (A,B)=(C,D) means that we have equilibrium; (A,B)>(C,D) means that the total weight of A and B is higher than the total weight of C and D; (A,B)<(C,D) means just the opposite.

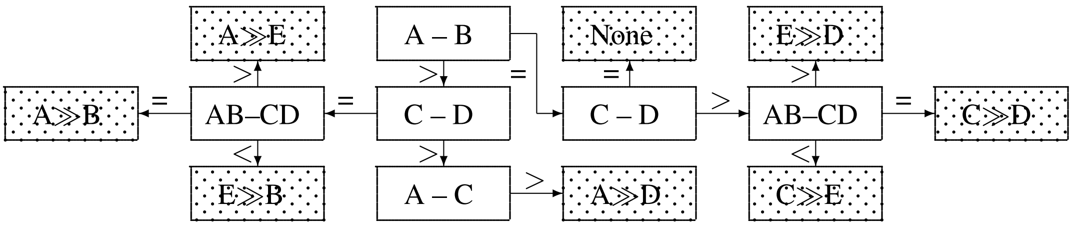
Problem 1.
All coins are normal except for possibly two, a heavy coin and a light coin whose total weight is equal to the total weight of two normal coins. Using at most four weighings, determine whether two such abnormal coins exist, and to identify them if they do. The total number of coins is
(a) 5; (b) 7; (c) 9.

Problem 2.
The total number of coins is at least 4. All are normal except for a heavy coin and a light coin. Using at most four weighings, determine whether the total weight of the two abnormal coins is
(1) greater than, (2) equal to, (3) less than
the total weight of two normal coins.

Solutions

Problem 1.

(a) We label the coins A, B, C, D and E. In the first two weighings, we compare A with B and C with D. If we have equilibrium both times, there are no abnormal coins. Suppose we do not have equilibrium either times. By symmetry, we may assume that A>B and C>D. In the third weighing, compare A with C. We cannot have equilibrium. By symmetry, we may assume that A>C. Then A is the heavy coin and D is the light coin. Suppose we have equilibrium exactly once in the first two weighings. By symmetry, we may assume that A>B and C=D. In the third weighing, compare (A,B) with (C,D). If (A,B)>(C,D), then A is the heavy coin and E is the light coin. If (A,B)=(C,D), then A is the heavy coin and B is the light coin. If (A,B)<(C,D), then E is the heavy coin and B is the light coin.



(b) We label the coins A, B, C, D, E, F and G. In the first three weighings, we compare A with B, C with D and E with F. We must have equilibrium at least once. If we have it all three times, there are no abnormal coins. Suppose we have equilibrium exactly twice. By symmetry, we may assume that A>B while C=D and E=F. In the fourth weighing, compare (A,B) with (C,D). If (A,B)>(C,D), then A is the heavy coin and G is the light coin. If (A,B)=(C,D), then A is the heavy coin and B is the light coin. If (A,B)<(C,D), then G is the heavy coin and B is the light coin. Suppose we have equilibrium exactly once. By symmetry, we may assume that A>B, C>D and E=F. In the fourth weighing, compare A with C. We cannot have equilibrium. By symmetry, we may assume that A>C. Then A is the heavy coin and D is the light coin.

(c) We label the coins A to I and arrange them in a 3×3 array.

A	B	C
D	E	F
G	H	I

The first weighing is top row against bottom row, comparing (A,B,C) with (G,H,I). The second weighing is left column against right column, comparing (A,D,G) with (C,F,I). The third weighing is middle row against middle column, comparing (D,F) with (B,H). The fourth weighing is up diagonal against down diagonal, comparing (C,G) with (A,I). After the first two weighing, there are three possible scenarios.

Scenario One. We have equilibrium both times.
Suppose the two abnormal coins exist. The first weighing tells us that they are in the same row, and the second weighing tells us that they are in the same column. Since this is impossible, there are no abnormal coins in this scenario.

Scenario Two. We do not have equilibrium either time.
By symmetry, we may assume that (A,B,C)>(G,H,I) and (A,D,G)>(C,F,I). Then A is the heavy coin while B, C, D and G are normal coins. In the third weighing, compare (A,I) with

(D,G). If (A,I)=(D,G), then I is the light coin. Otherwise, we go to the fourth weighing, comparing (D,F) with (B,H). If (D,F)=(B,H), then E is the light coin. Otherwise, whichever of F and H that is on the lighter side is the light coin.

Scenario Three. We have equlibrium exactly once.
By symmetry, we may assume that (A,B,C)>(G,H,I) and (A,D,G)=(C,F,I). Then the two abnormal coins exist and they are in the same column. The following chart summarizes the situations when the third and the fourth weighings are taken into consideration.

	(D,F)<(B,H)	(D,F)=(B,H)	(D,F)>(B,H)
(C,G)<(A,I)	A heavy, D light	A heavy, G light	D heavy, G light
(C,G)=(A,I)	B heavy E light	B heavy, H light	E heavy, H light
(C,G)>(A,I)	C heavy, F light	C heavy, I light	F heavy, I light

Problem 2.
The number of coins is $n \geq 4$. We first assume that $n = 7$. We label the coins A, B, C, D, E, F and G. In the first two weighings, compare A with B and C with D. Suppose we have equilibrium both times. In the third weighing, compare (B,C,D) with (E,F,G). The answer is Yes if and only if we have equilibrium. Suppose we have A>B and C>D. In the third weighing, compare (A,D) with (B,C). The answer is Yes if and only if we have equilibrium. Suppose we have equilibrium exactly once. By symmetry, we may assume that A>B and C=D. In the third weighing, compare E with F. If E=F, we compare (A,B,G) with (C,D,E) in the fourth weighing. The answer is Yes if and only if we have equilibrium. If E>F instead, we compare (A,F) with (B,E). The answer is Yes if and only if we have equilibrium.
Henceforth, we assume that $n \neq 7$. Divide the coins into four equal groups $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$ and $\mathscr{D}$, with no extra coin, one extra coin E, two extra coins E and F, and three extra coins E, F and G according to whether n is congruent modulo 4 to 0, 1, 2 and 3 respectively. In the first two weighings, compare $\mathscr{A}$ with $\mathscr{B}$ and $\mathscr{C}$ with $\mathscr{D}$. We consider three scenarios.

Scenario One. We have equilibrium both times.
Case 1. There are no extra coins.
The answer is (b) as both abnormal coins must belong to the same group.

Case 2. There is one extra coin E.
E must be normal and the answer is (b).

Case 3. There are two exta coins E and F.
They are either both normal or both abnormal. In the third weighing, compare E with F. If we have equilibrium, they are both normal, and the answer is (b). If not, we choose any C from $\mathscr{C}$ and any D from $\mathscr{D}$. In the fourth weighing, compare (C,D) with (E,F). The result gives us the answer.

Case 4. There are three exta coins E, F and G.
In the third weighing, compare E with F. If we have equilibrium, all extra coins are normal, and the answer is (b). If not, both abnormal coins are among the extra coins. Choose any C from $\mathscr{C}$ and any D_1 and D_2 from $\mathscr{D}$. (This is where we need $n \neq 7$.) In the fourth weighing, compare (C,D_1,D_2) with (E,F,G). The result will give us the answer.

Scenario Two. We do not have equilibrium either time.
All the extra coins are normal and may be ignored. By symmetry, we may assume that $\mathscr{A} > \mathscr{B}$ and $\mathscr{C} > \mathscr{D}$. In the third weighing, compare with $\mathscr{A}$ with $\mathscr{C}$. We cannot have equilibrium. By

symmetry, we may assume that $\mathscr{A} > \mathscr{C}$. Then the heavy coin is in $\mathscr{A}$ and the light coin is in $\mathscr{D}$. In the fourth weighing, compare ($\mathscr{A}$,$\mathscr{D}$) with ($\mathscr{B}$,$\mathscr{C}$). The result will give us the answer.

Scenario Three. We have equilibrium exactly once.
We may assume by symmetry that $\mathscr{A} > \mathscr{B}$ and $\mathscr{C} = \mathscr{D}$. Then all coins in $\mathscr{C}$ and $\mathscr{D}$ are normal.
Case 1. There are no extra coins.
The heavy coin is in $\mathscr{A}$ and the light coin is in $\mathscr{B}$. in the third weighing, we compare ($\mathscr{A}$,$\mathscr{B}$) with ($\mathscr{C}$,$\mathscr{D}$). The result will give us the answer.

Case 2. There is one extra coin E.
Choose any coin D in $\mathscr{D}$. In the third weighing, compare D with E. If D=E, then E is normal and we can handle this as in Case 1. If D>E, then E is the light coin and the heavy coin is in $\mathscr{A}$. In the fourth weighing, we compare ($\mathscr{A}$,E) with ($\mathscr{B}$,D). The result will give us the answer. If D<E, then E is the heavy coin and the light coin is in $\mathscr{B}$. In the fourth weighing, we compare ($\mathscr{A}$,D) with ($\mathscr{B}$,E). The result will give us the answer.

Case 3. There are two extra coins E and F.
Choose any C from $\mathscr{C}$ and any D from $\mathscr{D}$. In the third weighing, we compare (C,D) with (E,F). If we have equilibrium, both extra coins are normal and we can handle this as in Case 1. If (C,D)>(E,F), then E or F is the light coin and the heavy coin is in $\mathscr{A}$. In the fourth weighing, we compare ($\mathscr{A}$,E,F) with ($\mathscr{B}$,C,D). The result will give us the answer. If (C,D)<(E,F), then E or F is the heavy coin and the light coin is in $\mathscr{B}$. In the fourth weighing, we compare ($\mathscr{A}$,C,D) with ($\mathscr{B}$,E,F). The result will give us the answer.

Case 4. There are three extra coins E, F and G.
Choose any C from $\mathscr{C}$ and any D_1 and D_2 from $\mathscr{D}$. In the third weighing, we compare (C,D_1,D_2) with (E,F,G). If we have equilibrium, all extra coins are normal and we can handle this as in Case 1. If (C,D_1,D_2)>(E,F,G), we compare ($\mathscr{A}$,E,F,G) with ($\mathscr{B}$,C,D_1,D_2) in the fourth weighing. The result will give us the answer. If (C,D_1,D_2)<(E,F,G), we compare ($\mathscr{A}$,C,D_1,D_2) with ($\mathscr{B}$,E,F,G) in the fourth weighing. The result will give us the answer.

References

[1] Dick Hess, Problem 81, *Number-Crunching Math Puzzles*, Puzzle Wright Press, New York, 2009, pp. 34 and 84.

[2] Dick Hess, Problem 57, *Golf on the Moon*, Dover Publications Inc., Mineola, 2014, pp. 26–27 and 101.

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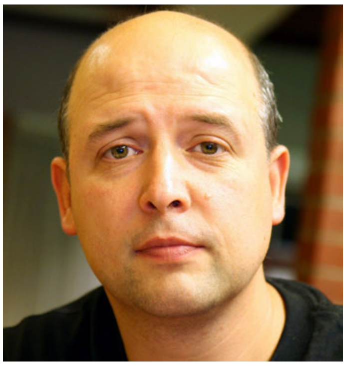
Economic Dissections into Equilateral Triangles

Dima Fomin, Boston and Andy Liu, Edmonton



Dmitri V. Fomin has worked in Applied Mathematics and Computer Science since 1994 as a research fellow in several software industry companies in Boston, USA. Prior to that career, in 1988-1994 he was a Senior Lecturer at the Chair of Higher Geometry and Topology of St.Petersburg State University, St.Petersburg, Russia. His areas of interest are geometry and topology, algebraic combinatorics, mathematical competitions, and history of mathematics. For several years in 1980-90s he was the Executive Secretary of the jury of Leningrad and St.Petersburg Mathematical Olympiads, member of the problem committee for the All-Union and All-Russia Math olympiads. He also contributed to the International Tournament of the Towns, participated in several conferences on extra-curricular mathematics and mathematical competitions. He published numerous books and articles on higher and elementary mathematics, math olympiads and problem solving, math education and its history, both in Russian and English.

Abstract
An equilateral triangle of side length 1 is cut off from a corner of an equilateral triangle of side length k for some positive integer k. The remaining part is to be dissected into as few equilateral triangles as possible. We study this minimum number $f(k)$.

As stated, $f(k)$ is the minimum number of equilateral triangles into which we dissect the remaining figure obtained by cutting off an equilateral triangle of side length 1 from a corner of an equilateral triangle of side length k for some positive integer k.

We have $f(1) = 0$ since nothing remains. In Figure 1, the first diagram shows that $f(2) = 3$, the second diagram that $f(4) = f(2^2) \leq 6$ and the third that $f(8) = f(2^3) \leq 9$. In general, $f(2^t) \leq 3t$.

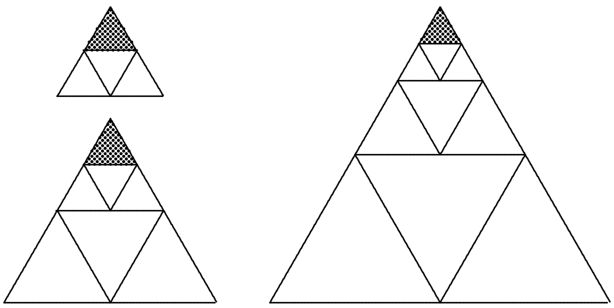


Figure 1

The first case not covered by the above construction is $k = 3$. We modify the construction for $k = 2$ by expanding the bottom right triangle, and completing the whole diagram by adding two more triangles, as showm in the first two diagrams in Figure 2. The modification from $k = 3$ to $k = 5$ is shown in the next two diagrams in Figure 2, and the modification from $k = 5$ to $k = 8$ is shown in the last two diagrams in Figure 2. We have $f(3) = f(F_4) = 5$, $f(5) = f(F_5) \leq 7$ and $f(8) = f(F_6) \leq 9$, where $\{F_t\}$ is the Fibonacci sequence $\{1, 1, 2, 3, 5, 8, \ldots\}$ defined by $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$. In general, $f(F_t) \leq 2t - 3$,

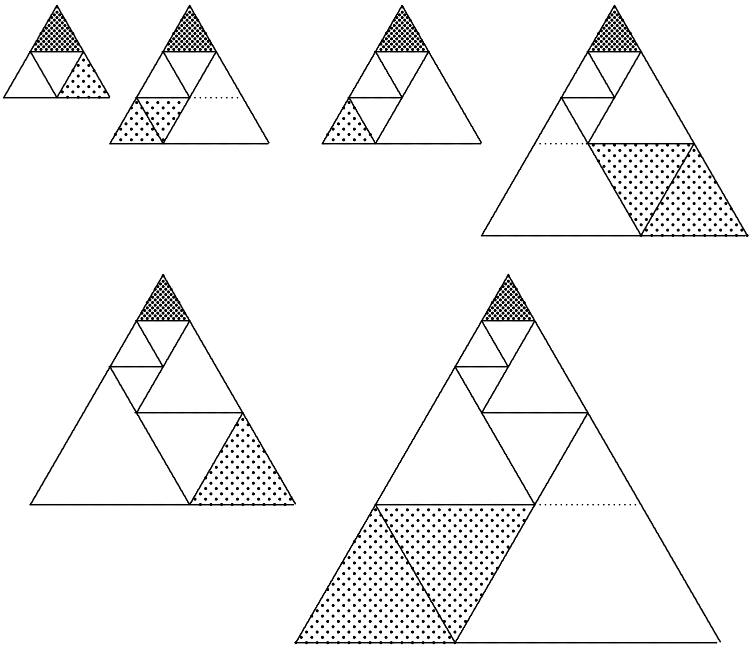


Figure 2

The first case not covered by either of the above constructions is $k = 6$. Figure 3 shows that $f(6) \leq 8$ and $f(7) \leq 9$.

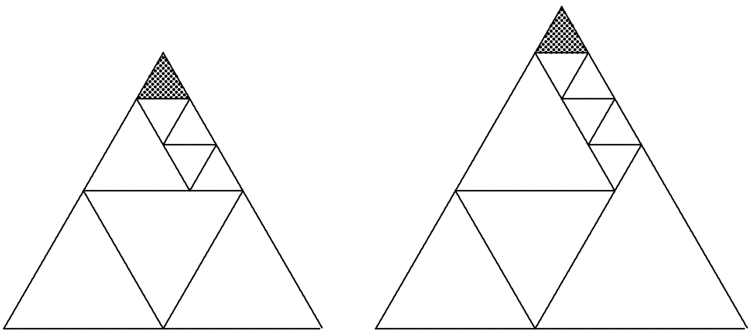


Figure 3

We now turn our attention to establishing lower bounds. Label the partition points along the left side of the original equilateral triangle $a_1 < a_2 < \cdots < a_m$, and the partition points along the right side $b_1 < b_2 < \cdots < b_n$. Their values are the numbers of rows the partition points below the top vertex. We always have $a_1 = b_1 = 1$ and $a_m = b_n = k$.

Each a_i, $1 \leq i < m$ and b_j, $1 \leq j < n$ is the top vertex of an equilateral triangle which is right side up, and one of the two top vertices of an equilateral triangle which is upside down. Note that if $a_i = b_j$, they may be associated with the same equaliteral triangle which is upside down.

Let $\{c_t\}$ be obtained by merging $\{a_i\}$ and $\{b_j\}$ into a single non-decreasing sequence, up to but excluding $a_m = b_n = k$. The number of equilateral triangles which are right side up is at least the number p of terms in $\{c_t\}$ and the number of equilateral triangles which are upside down is at least the number q of distinct terms in $\{c_t\}$. The total number of equilateral triangles is at least $p + q$.

Figure 4 shows a different construction that yields $f(6) \leq 8$. Here, $a_1 = b_1 = 1$, $a_2 = b_2 = 2$, $a_3 = 4$, $a_4 = b_3 = 6$ and $\{c_t\} = \{1, 1, 2, 2, 4\}$. We have $(p, q) = (5, 3)$ and $p + q = 8$.

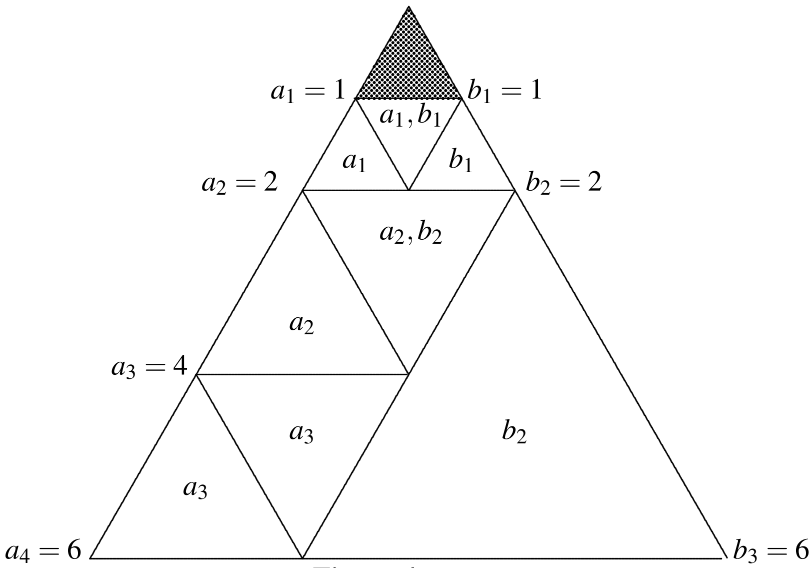


Figure 4

Figure 5 shows a different construction that yields $f(7) \leq 9$. Here, $a_1 = b_1 = 1$, $a_2 = 2$, $b_2 = 3$, $b_3 = 5$, $a_3 = b_4 = 7$ and $\{c_t\} = \{1, 1, 2, 3, 5\}$. We have $(p, q) = (5, 4)$ and $p + q = 9$.

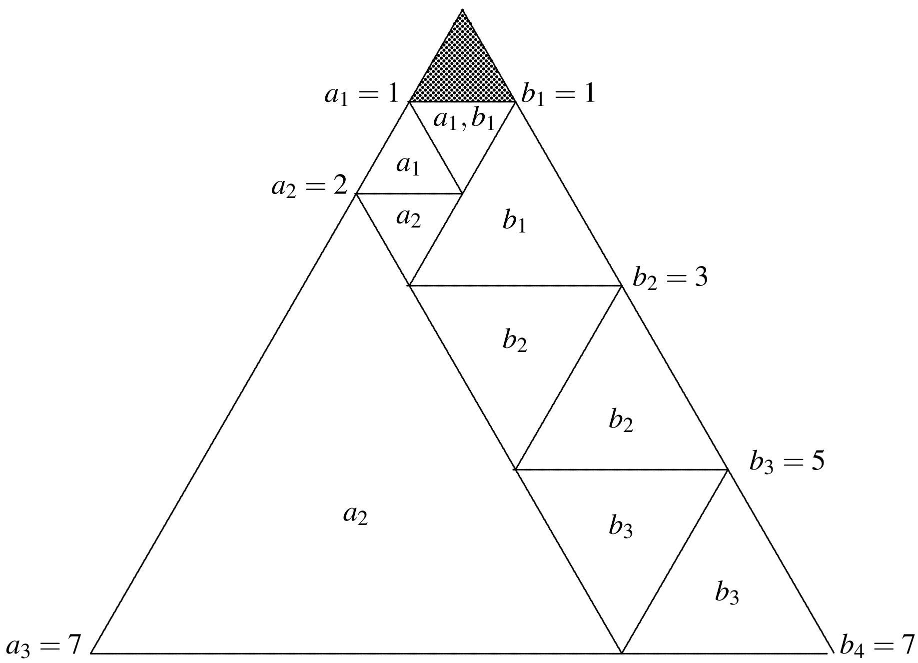


Figure 5

We now come to the key idea, that $\{c_t\}$ cannot grow faster than the Fibonacci sequence.

Lemma. $c_t + c_{t+1} \geq c_{t+2}$.

Proof:
We consider two cases.
Case 1. c_t and c_{t+1} belong to different sequences.
Let $c_t = a_i$ and $c_{t+1} = b_j$ for some i and j. Let ℓ be an integer such that $a_i < \ell \leq a_{i+1}$ and $b_j < \ell \leq b_{j+1}$. The horizontal line ℓ rows below the top vertex intersects the $(i+1)$st triangle on the left and the $(j+1)$st triangle on the right. Since these two triangles cannot overlap, the sum of their side lengths is at most ℓ. Hence $(\ell - a_i) + (\ell - b_j) \leq \ell$ or, equivalently, $a_i + b_j \geq \ell$. It follows

that $a_i + b_j \geq \min\{a_{i+1}, b_{j+1}\}$, which is equivalent to the desired result.

Case 2. c_t and c_{t+1} belong to the same sequence.
We may take $c_t = a_i$ and $c_{t+1} = a_{i+1}$ for some i. There are two subcases.
Subcase 2(a). $c_{t+2} = b_{j+1}$ for some j.
We have $b_{j+1} \geq a_{i+1} > a_i \geq b_j$. Suppose that $b_{j+1} > a_i + a_{i+1}$. Then $b_{j+1} - b_j \geq b_{j+1} - a_i > a_{i+1}$. Figure 6 on the left shows that we must have $a_{i+2} < b_{j+1}$. This means that c_{t+2} is equal to a_{i+2} and not b_{j+1}. We have a contradiction.
Subcase 2(b). $c_{t+2} = a_{i+2}$.
We have $b_{j+1} \geq a_{i+2} > a_{i+1} > a_i > b_j$. Suppose that $a_{i+2} > a_i + a_{i+1}$. Then $a_{i+2} - a_{i+1} > a_i$. Figure 6 on the right shows that we must have $b_{j+1} < a_{i+2}$. This means that c_{t+2} is equal to b_{j+1} and not a_{i+2}. We have a contradiction.

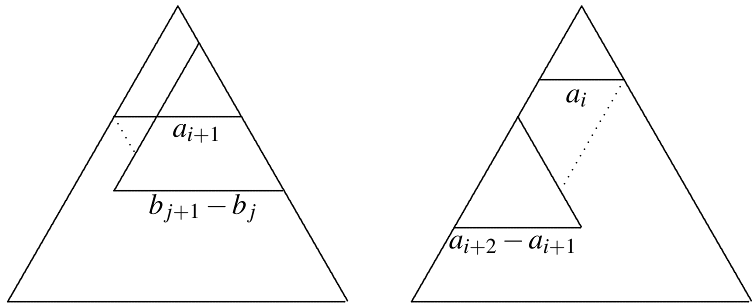


Figure 6

In the optimal dissection, we have $f(k) = p + q$. Consider $f(F_t)$ for $t \geq 2$. We have $p \geq t-1$, and since the Fibonacci sequence has no duplicates other than $F_1 = 1 = F_2$, $q \geq t-2$ so that $f(F_t) \geq 2t-3$. On the other hand, it is not clear whether we do have $f(2^t) \geq 3t$ for all t.

We conclude our discussion with a numerical example, namely, $f(23) = 15$. Figure 7 shows an upper bound based on the sequence $\{c_t\} = \{1, 1, 2, 3, 4, 5, 9, 14\}$.

For the lower bound, note that $F_8 = 21 < 23 < 34 = F_9$. Hence we must have $p \geq 8$. Since the Fibonacci sequence does not contain three equal terms, $q \geq \frac{p}{2}$. Hence if $p \geq 10$, we already have $p + q \geq 15$, and cannot have $f(23) \leq 14$.

Suppose that $p = 9$. In order to have $f(23) \leq 14$, we must have $q = 5$. Hence $\{c_t\}$ must have four pairs of equal terms. Since 23 is odd, $c_8 \neq c_9$. The situation is impossible.

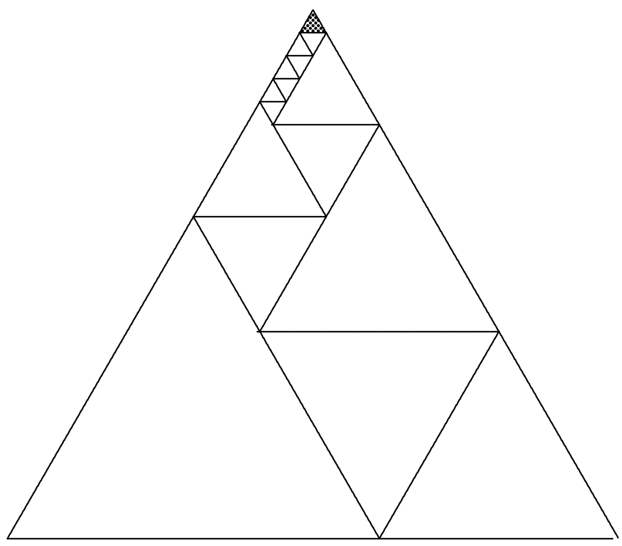


Figure 7

Finally, suppose that $p = 8$. In order to have $f(23) \leq 14$, we must have $q \leq 6$. Hence $\{c_t\}$ has at least a pair of equal terms other than $c_1 = c_2 = 1$. Then the common value of these two terms, which is greater than 1, must divide 23. We have a contradiction.

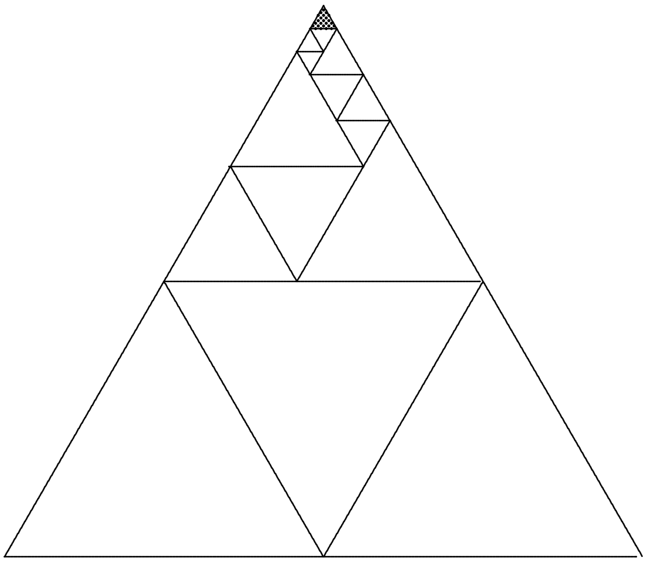


Figure 8

On the other hand, as is shown in Figure 8, $f(24) \leq 14$. This is the first of many breaks in monotonicity, so that a general formula for $f(k)$ is extremely unlikely.

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The 63rd International Mathematical Olympiad

Angelo Di Pasquale
IMO Team Leader, Australia

Angelo was twice a contestant at the International Mathematical Olympiad. He completed a PhD in mathematics at the University of Melbourne studying algebraic curves. He is currently Director of Training for the Australian Mathematical Olympiad Committee (AMOC), and Australian Team Leader at the International Mathematical Olympiad.
He enjoys composing Olympiad problems for mathematics contests.

The 63rd International Mathematical Olympiad (IMO) was held 6–16 July 2022 in the city of Oslo, Norway. This was the first time that Norway has hosted the IMO.

A total of 589 high school students from 104 countries participated. Of these, 68 were female.

As per normal IMO rules, each participating country may enter a team of up to six students, a Team Leader and a Deputy Team Leader.[1]

Participating countries also submit problem proposals for the IMO. This year there were 193 problem proposals from 50 countries. The local Problem Selection Committee shortlisted 33 of these for potential use on the IMO exams.

At the IMO the Team Leaders, as an international collective, form what is called the *Jury*. The Jury makes the important decisions that shape each year's IMO. Their first task is to set the two IMO competition papers from the aforementioned shortlist and approve marking schemes. During this period the Leaders and their observers are trusted to keep all information about the contest problems completely confidential.

The six problems that ultimately appeared on the IMO exam papers may be described as follows.

1. An easy combinatorics problem proposed by France.
2. A functional inequality proposed by the Netherlands. Although intended to be a medium-level problem, it turned out to be easier than the Jury anticipated.
3. A beautiful difficult number theory problem proposed by the United States of America.
4. An easy classical geometry problem proposed by Slovakia.
5. A medium to easy number theory problem proposed by Belgium.
6. A beautiful difficult combinatorics problem proposed by Serbia.

[1] The IMO regulations also permit countries to enter a small number of additional staff as Observers. These may fulfil various roles such as meeting child safety obligations, assisting with marking and coordination, or learning about how to host an IMO.

These six problems were posed in two exams held on Monday 11 July and Tuesday 12 July. Each exam paper had three problems. The contestants worked individually. They were allowed four and a half hours per paper to attempt the problems. Each problem was scored out of a maximum of seven points.

After the exams, the Leaders and their Deputies spent about two days assessing the work of the students from their own countries, guided by the marking schemes which had been agreed to earlier. A local team of markers called Coordinators also assessed the papers. They too were guided by the marking schemes but are allowed some flexibility if, for example, a Leader brought something to their attention in a contestant's exam script that is not covered by the marking scheme. The Team Leader and Coordinators must agree on scores for each student of the Leader's country in order to finalise scores. Any disagreements that cannot be resolved in this way are ultimately referred to the Jury. No such referrals occurred this year.

The contestants found Problems 1 and 4 to be the easiest with average scores of 5.54 and 5.47, respectively. Problem 6 was the hardest, averaging just 0.68. The score distributions by problem number were as follows.

Mark	**P1**	**P2**	**P3**	**P4**	**P5**	**P6**
0	34	104	407	74	112	434
1	56	89	68	31	115	80
2	19	23	69	18	44	22
3	21	41	6	17	48	7
4	10	7	3	10	15	19
5	13	3	4	5	35	3
6	51	19	4	1	49	2
7	385	303	28	433	171	22
Mean	5.54	4.31	0.81	5.47	3.52	0.68

The medal cuts were set at 34 points for Gold, 29 for Silver and 23 for Bronze. The medal distributions[2] were as follows.

	Gold	**Silver**	**Bronze**	**Total**
Number	44	101	140	285
Proportion	7.5%	17.1%	23.8%	48.4%

These awards were presented at the closing ceremony. Of those who did not get a medal, a further 210 contestants received an Honourable Mention for scoring full marks on at least one problem.

Ten contestants achieved the most excellent feat of a perfect score of 42. This included all six team members from the People's Republic of China.[3]

The 2022 IMO was organised by Norway's Ministry of Education and The University of Oslo.

Hosts for future IMOs have been secured as follows.

2–13 July, 2023	Chiba, Japan
2024	United Kingdom
2025	Melbourne, Australia

[2]The total number of medals is approved by the Jury and should not normally exceed half the total number of contestants. The numbers of Gold, Silver and Bronze medals should be approximately in the ratio 1:2:3.

[3]It is exceedingly difficult for an entire team of six to achieve a perfect score. The only other time this has occurred was when the United States of America achieved this at the 1994 IMO.

Much of the statistical information found in this report can also be found on the official website of the IMO.

www.imo-official.org

OSLO2022

English (eng), day 1

Monday, 11. July 2022

Problem 1. The Bank of Oslo issues two types of coin: aluminium (denoted A) and bronze (denoted B). Marianne has n aluminium coins and n bronze coins, arranged in a row in some arbitrary initial order. A *chain* is any subsequence of consecutive coins of the same type. Given a fixed positive integer $k \leqslant 2n$, Marianne repeatedly performs the following operation: she identifies the longest chain containing the k^{th} coin from the left, and moves all coins in that chain to the left end of the row. For example, if $n = 4$ and $k = 4$, the process starting from the ordering $AABBBABA$ would be

$$AAB\underline{B}BABA \to BBB\underline{A}AABA \to AAA\underline{B}BBBA \to BBB\underline{B}AAAA \to BBB\underline{B}AAAA \to \cdots.$$

Find all pairs (n,k) with $1 \leqslant k \leqslant 2n$ such that for every initial ordering, at some moment during the process, the leftmost n coins will all be of the same type.

Problem 2. Let $\mathbb{R}^+$ denote the set of positive real numbers. Find all functions $f\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that for each $x \in \mathbb{R}^+$, there is exactly one $y \in \mathbb{R}^+$ satisfying

$$xf(y) + yf(x) \leqslant 2.$$

Problem 3. Let k be a positive integer and let S be a finite set of odd prime numbers. Prove that there is at most one way (up to rotation and reflection) to place the elements of S around a circle such that the product of any two neighbours is of the form $x^2 + x + k$ for some positive integer x.

Language: English

Time: 4 hours and 30 minutes.
Each problem is worth 7 points.

English (eng), day 2

Tuesday, 12. July 2022

Problem 4. Let $ABCDE$ be a convex pentagon such that $BC = DE$. Assume that there is a point T inside $ABCDE$ with $TB = TD$, $TC = TE$ and $\angle ABT = \angle TEA$. Let line AB intersect lines CD and CT at points P and Q, respectively. Assume that the points P, B, A, Q occur on their line in that order. Let line AE intersect lines CD and DT at points R and S, respectively. Assume that the points R, E, A, S occur on their line in that order. Prove that the points P, S, Q, R lie on a circle.

Problem 5. Find all triples (a, b, p) of positive integers with p prime and

$$a^p = b! + p.$$

Problem 6. Let n be a positive integer. A *Nordic square* is an $n \times n$ board containing all the integers from 1 to n^2 so that each cell contains exactly one number. Two different cells are considered adjacent if they share a common side. Every cell that is adjacent only to cells containing larger numbers is called a *valley*. An *uphill path* is a sequence of one or more cells such that:

(i) the first cell in the sequence is a valley,

(ii) each subsequent cell in the sequence is adjacent to the previous cell, and

(iii) the numbers written in the cells in the sequence are in increasing order.

Find, as a function of n, the smallest possible total number of uphill paths in a Nordic square.

Language: English

Time: 4 hours and 30 minutes.
Each problem is worth 7 points.

Some IMO Country Totals

Rank	Country	Total
1	People's Republic of China	252
2	Republic of Korea	208
3	United States of America	207
4	Vietnam	196
5	Romania	194
6	Thailand	193
7	Germany	192
8	Islamic Republic of Iran	191
8	Japan	191
10	Israel	188
10	Italy	188
12	Poland	183
13	United Kingdom	179
14	Canada	178
14	Taiwan	178
16	Bulgaria	177
17	Kazakhstan	174
17	Ukraine	174
19	Brazil	173
19	Hong Kong	173
19	Peru	173
22	Saudi Arabia	168
23	Mexico	167
24	India	165
24	Singapore	165
26	Armenia	163
26	Greece	163
26	Turkey	163
29	Australia	162
29	Mongolia	162

Distribution of Awards at the 2022 IMO

Country	Total	Gold	Silver	Bronze	HM
Albania	62	0	0	0	3
Algeria	75	0	0	1	3
Argentina	139	0	1	3	2
Armenia	163	0	2	4	0
Australia	162	1	0	4	1
Austria	118	0	0	2	4
Azerbaijan	157	0	1	4	1
Bangladesh	115	0	0	1	5
Belarus	160	0	2	3	1
Belgium	94	0	0	1	5
Bolivia	57	0	0	0	4
Bosnia and Herzegovina	154	0	3	2	1
Botswana	17	0	0	0	0
Brazil	173	2	1	2	1
Bulgaria	177	1	3	1	1
Canada	178	2	2	1	1
Chile	27	0	0	0	1
Colombia	74	0	0	1	2
Costa Rica	92	0	0	1	4
Croatia	157	0	1	4	1
Cuba	13	0	0	0	1
Cyprus	85	0	0	0	5
Czech Republic	122	0	1	2	3
Denmark	121	0	0	2	4
Dominican Republic	26	0	0	0	2
Ecuador	27	0	0	0	2
El Salvador	56	0	1	0	2
Estonia	124	0	1	2	3
Finland	101	0	0	1	4
France	158	0	3	2	1
Georgia	121	0	0	3	3
Germany	192	1	4	1	0
Ghana	18	0	0	0	0
Greece	163	0	2	3	1
Honduras	31	0	0	0	2
Hong Kong	173	1	3	1	1
Hungary	158	0	1	5	0
Iceland	51	0	0	0	3
India	165	1	0	5	0
Indonesia	151	0	1	4	1

Country	Total	Gold	Silver	Bronze	HM
Ireland	84	0	0	0	5
Islamic Republic of Iran	191	3	3	0	0
Israel	188	1	4	1	0
Italy	188	2	2	2	0
Japan	191	1	4	1	0
Kazakhstan	174	0	3	3	0
Kosovo	55	0	0	0	4
Kyrgyzstan	73	0	0	0	4
Latvia	94	0	0	0	5
Liechtenstein	12	0	0	0	1
Lithuania	129	0	1	1	4
Luxembourg	26	0	0	0	1
Macau	127	0	0	3	3
Malaysia	137	1	1	1	3
Mauritania	42	0	0	0	4
Mexico	167	0	2	4	0
Mongolia	162	0	2	3	1
Montenegro	23	0	0	0	1
Morocco	82	0	0	0	4
Nepal	27	0	0	0	2
Netherlands	145	1	0	2	3
New Zealand	108	0	1	0	5
North Macedonia	148	0	2	2	2
Norway	103	0	0	1	4
Oman	19	0	0	0	0
Pakistan	54	0	0	0	3
Palestine	14	0	0	0	1
Panama	42	0	0	0	3
Paraguay	59	0	0	0	4
People’s Republic of China	252	6	0	0	0
Peru	173	0	3	3	0
Philippines	119	0	0	2	4
Poland	183	0	4	2	0
Portugal	137	0	1	3	2
Puerto Rico	40	0	0	0	1
Republic of Korea	208	3	3	0	0
Republic of Moldova	135	0	1	2	3
Romania	194	2	4	0	0
Rwanda	34	0	0	0	1
Saudi Arabia	168	0	2	4	0
Serbia	153	0	2	2	2
Singapore	165	0	4	1	1

Country	Total	Gold	Silver	Bronze	HM
Slovakia	136	0	1	4	1
Slovenia	79	0	0	1	3
South Africa	94	0	0	0	6
Spain	139	0	0	4	2
Sri Lanka	77	0	0	0	5
Sweden	106	0	0	1	5
Switzerland	145	0	0	4	2
Syria	82	0	0	1	5
Taiwan	178	1	2	3	0
Tajikistan	107	0	0	1	5
Thailand	193	3	2	1	0
Tunisia	85	0	0	0	6
Turkey	163	0	4	1	1
Uganda	13	0	0	0	1
Ukraine	174	1	1	4	0
United Arab Emirates	10	0	0	0	1
United Kingdom	179	1	3	2	0
United States of America	207	4	1	1	0
Uruguay	27	0	0	0	2
Uzbekistan	124	0	0	1	5
Venezuela	7	0	0	0	0
Vietnam	196	2	2	2	0
Total (104 teams, 589 contestants)		**44**	**101**	**140**	**210**

N.B. Not all countries entered a full team of six students.

Angelo Di Pasquale
IMO Team Leader, Australia

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International Mathematics Tournament of the Towns

Andy Liu

In 1976, Andy Liu received a Doctor of Philosophy in mathematics and a Professional Diploma in elementary education, making him one of very few people officially qualified to teach from kindergarten to graduate school. He was heavily involved in the International Mathematical Olympiad. He served as the deputy leader of the USA team from 1981 to 1984, and as the leader of the Canadian team in 2000 and 2003. He chaired the Problem Committee in 1995, and was on the Problem Committee in 1994, 1998 and 2016. He had given lectures to school children in Canada, the United States, Colombia, Hungary, Latvia, Sweden, Tunisia, South Africa, Sri Lanka, Nepal, Thailand, Laos, Malaysia, Indonesia, the Philippines, Hong Kong, Macau, Taiwan, China and Australia. He ran a mathematics circle in Edmonton for thirty-two years, and continued his book-publishing after his retirement from the University of Alberta in 2013. He is currently writing his twentieth mathematics book, which is based on Greek Mythology.

Selected Problems from the Spring 2023 Papers

1. There is a single coin on each square of a 5×5 board. All but two are normal coins of equal weight. The two abnormal coins have the same weight as each other but lighter than normal coins. The two abnormal coins occupy squares sharing exactly one vertice. Is it possible to determine for sure
   (a) 13;
   (b) 15;
   (c) 17

   normal coins in a single weighing on a balance?

2. Let I be the incenter of triangle ABC. Let the bisector of $\angle B$ intersect CA at E. The tangent to the circumcircle of triangle AIE at A and the tangent to the circumcircle of triangle CIE at C intersect at point T. Prove that AC is perpendicular to IT.

3. Prove that an angle bisector of a right triangle with a 30° angle is twice as long as another of its angle bisectors.

4. There is an amoeba on one square of a 10×10 board. In the first move, the amoeba moves to an adjacent square in the same row or column and splits into two amoebas, both staying in that square. In each move after the first, one of the amoebas makes a similar move-and-split. Is it possible that after a number of moves, the number of amoebas in each square is the same?

5. The digits of a positive integer consist of m ones and some zeros. The digits of another positive integer consist of n ones and some zeros. The digits of their product consist of k ones and some zeros. Is it necessary true that $k = mn$?

6. Of the positive integers from 1 to 100, 50 are painted in red, 25 in yellow and 25 in green. The red and yellow integers can be divided into 25 triples such that in each triple, the colors of the numbers in ascending order are red-yellow-red. The red and green integers can be

divided into 25 triples such that in each triple, the colors of the numbers in ascending order are red-green-red. Is it always possible to divide the 100 integers into 25 quadruples such that in each quadruple, the colors of the numbers in ascending order are red-yellow-green-red or red-green-yellow-red?

7. A set of integers can be partitioned into n disjoint non-constant arithmetic progressions, infinite in both directions, but cannot be partitioned into a smaller number of such progressions. Is such a partition always unique if
   (a) $n = 2$;
   (b) $n = 3$?

8. Given are two binary sequences of lengths 100. In a move, we can either insert a block of any number of zeros, or a block of any number of ones, anywhere in the sequence, including at the beginning or at the end, or remove such a block from a sequence. Prove that it is possible to transform the first sequence into the second one in at most 100 moves.

9. The perimeter of triangle ABC equals 1. The excircle of ABC opposite A touches the extension of AB at P and the extension of AC at Q. The line passing through the midpoints of AB and AC intersects the circumcircle of triangle APQ at X and Y. Determine the length of XY.

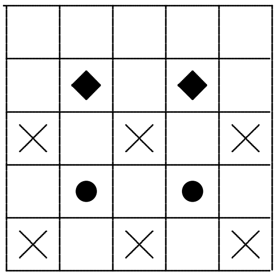
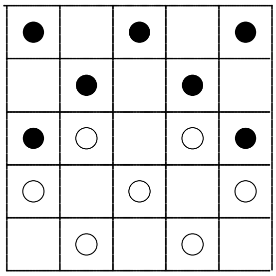
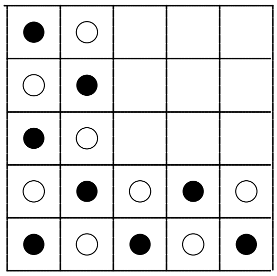
10. Let $P(x)$ be a polynomial of degree $n > 5$ with integer coefficients and with n distinct integer roots. Prove that $P(x) + 3$ has n distinct real roots.

11. Let $n > 1$ be an integer. Every fraction with denominator less than n can be expressed as a combination of additions and subtractions of fractions whose numerators and denominators add up to n. The fractions may be reducible. Prove that n has this property if and only if n is prime.

12. Chameleons come in five colors. If a chameleon bites another one, the color of the bitten chameleon changes into one of these colors according to some set of rules. Each set consists of 25 expressions of the new color of the bitten chameleon as a function of the color of the biting chameleon and the old color of the bitten chameleon. There are 2023 red chameleons which can all become blue eventually by biting only one another, following some sets of rules. What is the fewest number of red chameleons which can all become blue eventually by biting only one another, following every possible set of rules used by the 2023 red chameleons?

Solutions

1. Paint the board in the usual checkered pattern with black corner squares. We put black coins on black squares and white coins on white squares. Both abnormal coins have the same color.
   (a) We weigh the diamond coins against the circular coins in the diagram below on the left. If we have equilibrium, all black coins are normal, and we have identified 13 of them. Otherwise, we may assume by symmetry that the diamond coins are heavier. One of the circular coins is abnormal. The other abnormal coin only occupy a square marked with a cross in the diagram below on the left. Hence we have identified $25 - 2 - 6 = 17$ normal coins. Overall, we can identify 13 normal coins.

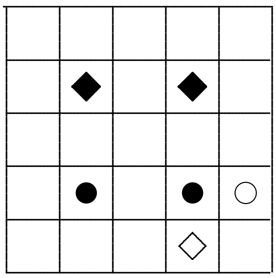
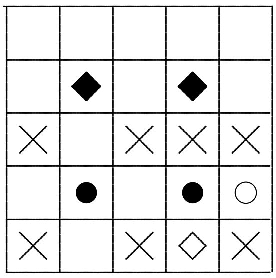
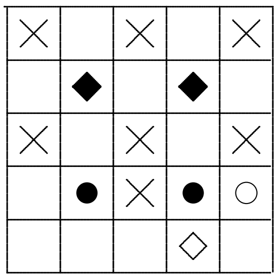
  

The result may be improved to 14 by weighing the black coins against the white coins in the diagram above in the middle or on the right. We omit the details.

(b) We present three different solutions.

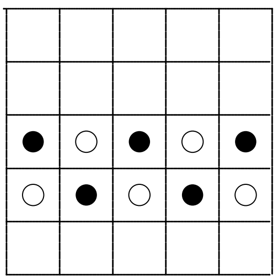
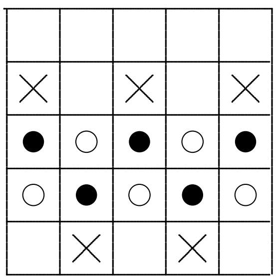
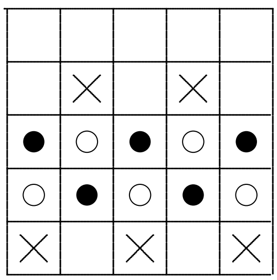
First Solution:

We weigh the diamond coins against the circular coins in the diagram below on the left. If we have equilibrium, all black coins along the with two white coins being weighed are normal, and we have identified 15 of them. If the diamond coins are heavier, the abnormal coin not being weighed can only occupy a square marked with a cross in the diagram below in the middle. Hence we have identified $25-3-7=15$ normal coins. If the circular coins are heavier, the situation is shown in the diagram below on the right, with the same result.

Second Solution:

We weigh the black coins against the white coins in the diagram below on the left. If we have equilibrium, the 15 coins in the bottom three rows are normal. If the black coins are heavier, an abnormal coin not being weighed can only occupy a square marked with a cross in the diagram below in the middle. Hence we have identified $25-5-5=15$ normal coins. If the white coins are heavier, the situation is shown in the diagram below on the right, with the same result.

Third Solution:

We weigh the black coins against the white coins in the diagram below on the left. If we have equilibrium, the 16 coins being weighed are normal. If the black coins are heavier, an abnormal coin not being weighed can only occupy a square marked with a cross in the diagram below in the middle. Hence we have identified $25-8-2=15$

normal coins. If the white coins are heavier, the situation is shown in the diagram below on the right, with the same result.

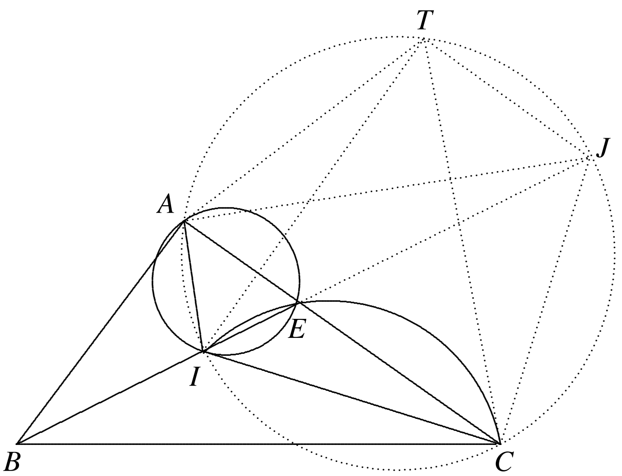
●	○	●		●
○	●		●	
●	○	●		●
○		○		○
	○		○	

●	○	●		●
○	●	×	●	
●	○	●	×	●
○		○		○
	○		○	

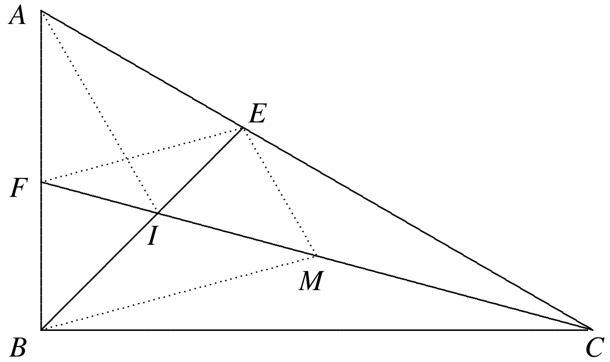
●	○	●		●
○	●		●	
●	○	●		●
○	×	○	×	○
	○		○	

(c) The task is impossible. Suppose to the contrary that it is possible. In one weighing, there are three possible outcomes. In each outcome, if we can identify at least 17 normal coins, that would leave at most $25 - 17 = 8$ coins as possibly abnormal. Overall, at most $8 \times 3 = 24$ of the 25 may be abnormal, meaning that there is one particular coin which must always be normal. This is clearly not the case, and we have a contradiction.

2. We assume that $AB \neq BC$. Let J be the excenter of triangle ABC opposite B. Then J lies on the extension of BI. We have $\angle IAJ = 90^\circ = \angle ICJ$, so that A and C lie on the circle with diameter IJ. Hence $\angle CAJ = \angle CIJ = \angle ACT$ and $\angle ACJ = \angle AIJ = \angle CAT$. It follows that triangles ACT and CAT are congruent, so that TJ is parallel to AC. Moreover, T also lies on the circle with diameter IJ, so that $\angle ITJ = 90^\circ$. Hence IT is perpendicular to TJ, and hence also to AC. If $AB = BC$, then T and J coincide and the desired result is trivial.



3. Let I be the incenter of triangle ABC and M be the midpoint of CF. Then $CF = 2BM$. We claim that $BM = BE$. We have $\angle AEB = \angle ACB + \angle CBE = 75^\circ$. We also have $\angle BFC = 90^\circ - \frac{1}{2}\angle BCA = 75^\circ$. Hence $AEIF$ is cyclic, so that $\angle BEF = \frac{1}{2}\angle CAB = 30^\circ$. Since $\angle BMF = 160^\circ - \angle BFM - \angle FBM = 30^\circ$, $BMEF$ is also cyclic. It follows that $\angle BEM = \angle BFM = 75^\circ$. We have $\angle MBE = \angle MBC + \angle MCB = 30^\circ$, so that $\angle BME = 75^\circ$. Hence $BE = BM$, justifying our claim.



4. Paint the square of the board in the usual checkered pattern. Let d_n be the difference between the number of amoebas on white squares and the number of amoebas on black squares after n moves. Assuming that the initial amoeba is on a white square. Then $d_0 = 1$ and $d_1 = -2$. In order for every square to have the same number of amoebas, a necessary condition is $d_n = 0$ for some n, However, $d_{n+1} = d_n \pm 3$. Hence $d_n \equiv 1 \pmod 3$ for all n, and can never be 0. Thus the desired situation is impossible.

5. We can construct a counter-example with $m = n = 10$ but $k = 91$. The multiplicand consists of 10 ones separated by sufficiently large amounts of zeros. Ten copies of this number are lined up in ten rows so that the ith one in the ith row are in the same column for $1 \leq i \leq 10$. By adjusting the number of zeros, we can make sure that no other column contains two or more ones. The rows then determine the multiplier, which also consists of ten ones and a large number of zeros. When the product is computed, a carrying occurs and the number of ones is reduced from 100 to 91.

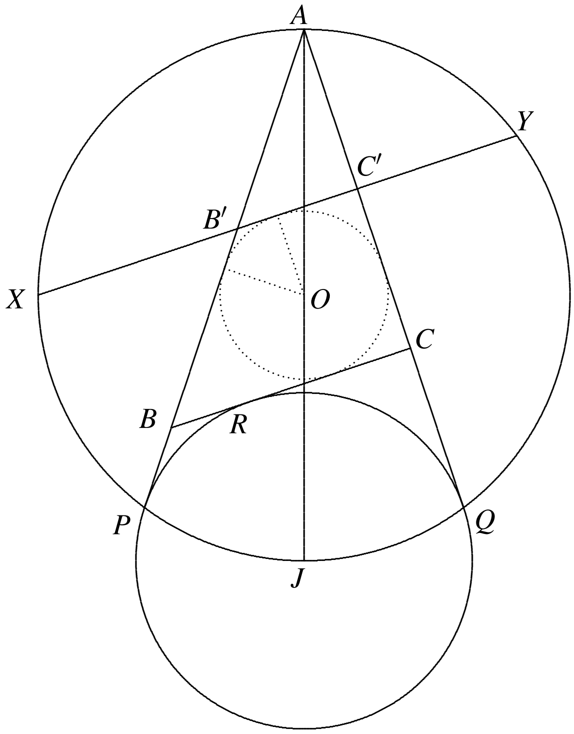
6. Let the yellow numbers be $y_1 < y_2 < \cdots < y_{25}$, the green numbers be $g_1 < g_2 < \cdots < g_{25}$ and the red numbers be $r_1 < r_2 < \cdots < r_{25} < R_1 < R_2 < \cdots < R_{25}$. We claim that for $1 \leq k \leq 25$, (r_k, y_k, g_k, R_k) is a quadruple with the desired properties. Since each of y_1, y_2, $\ldots$, y_k is greater than a distinct red number, we have $r_k < y_k$. Similarly, $y_k < R_k$, $g_k > r_k$ and $g_k < R_k$, justifying our claim.

7. (a) Suppose we have four arithmetic progressions A, B, C and D, with non-zero common differences a, b, c and d respectively, such that $A \cup B = C \cup D$ and $A \cap B = \emptyset = C \cap D$. We may assume that $a = \min\{a, b, c, d\}$. If $A \neq C$ and $A \neq D$, then each of $A \cap C$ and $A \cap D$ is an arithmetic progression, with common difference divisible by a. Hence $A \supset C$ and $A \supset D$. Then $C \supset B$ and $D \supset B$, so that $A \supset B$. Since $A \cap B = \emptyset$, $B = \emptyset$ so that the whole set of numbers is a single arithmetic progression. This is a contradiction, and the partition into two arithmetic progressions must be unique.

   (b) Take all the integers modulo 12 in residue classes 0, 3, 4, 6, 8 and 9. If we form an arithmetic progression with common difference 3, using classes 0, 3, 6 and 9, we have to take classes 4 and 8 as two separate arithmetic progressions with common differences 12. If we form an arithmetic progression with common difference 4, using classes 0, 4 and 8, we can form another arithemetic progression with common difference 6, using classes 3 and 9. Then we have to take class 8 as a separate arithmetic progression with common difference 12. Hence there are no ways of forming only two arithmetic progreesions but two ways of forming three arithmetic progressions.

8. More generally, we prove by mathematical induction on n that the transformation between any two binary sequences of length n can be accomplished in n moves for all $n \geq 2$. The verification of the base case $n = 2$ is routine. Assume that the result holds for some $n \geq 2$. Consider any two binary sequences of length $n+1$. Suppose they agree in at least one term, say the $(k+1)$st one. By the induction hypothesis, the first k terms can be taken care of in k moves while the remaining $(n+1)-(k+1)$ terms can be taken care of in $n-k$ moves, for a total of only n moves. Henceforth, we suppose that the two sequences do not agree in any term. If one of them consists only of 0s and the other consists only of 1s, two moves are sufficient. If not, then the first sequence contains two consecutive terms which are unequal, say 0 as the kth term and 1 as the $(k+1)$st term. These values are interchanged for the second sequence. They can be taken care of in two moves, and we can complete the argument by applying the induction hypothesis as before.

9. Let O be the circumcenter of triangle APQ. Let B' be the midpoint of AB and C' be the midpoint of AC. Let R be the point of tangency of BC with the excircle of triangle ABC opposite A.



Use a homothety with center A and ratio $\frac{1}{2}$ to contract B to B', C to C' and the excircle of triangle ABC opposite A to the excircle of triangle $AB'C'$ opposite A. The center of this circle is O. Since it is tangent to both AP and XY, we have $XY = AP$. Now $XY = \frac{1}{2}$ since $1 = BC+CA+AB = (RC+CA)+(AB+BR) = (QC+CA)+(AB+BP) = 2AP$.

10. Let $x_1 < x_2 < \cdots < x_n$ be the integer roots of $P(x)$. Then $P(x) = a(x-x_1)(x-x_2)\cdots(x-x_n)$ for some $a \neq 0$. The real line is divided into $n+1$ intervals $(-\infty, x_1)$, $x_1, x_2)$, (x_2, x_3), $\ldots$, (x_{n-1}, x_n) and (x_n, ∞). On these intervals, $P(x)$ is alternately positive and negative. For $1 \le i \le n-1$, let $y_i = \frac{x_1 + x_{i+1}}{2}$. Then it is half of an integer. Hence the minimum absolute value of each factor in $P(x)$ is $\frac{1}{2}$. There may be two copies of $\frac{1}{2}$, and as we shift away to adjacent factors, this absolute value increases by at least 1. Since $n > 5$, we have $|P(y_i)| \ge (\frac{1}{2})^2(\frac{3}{2})^2(\frac{5}{2})^2 > 3$. Define $Q(x) = P(x) + 3$. We consider two cases.
Case 1. n is even.
If $a > 0$, then $P(x) < 0$ on (x_1, x_2), (x_3, x_4), $\ldots$, (x_{n-1}, x_n). On each of them, $Q(x) = 3$ at the endpoints and $Q(x) < 0$ at the midpoint. Hence each of these $\frac{n}{2}$ intervals contains two distinct real roots of $Q(x)$, bringing the total to n. If $a < 0$, then $P(x) < 0$ on $(-\infty, x_1)$, (x_2, x_3), (x_4, x_5), $\ldots$, (x_{n-2}, x_{n-1}) and (x_n, ∞). As before, each of the $\frac{n}{2} - 1$ finite intervals contains two distinct real roots of $Q(x)$. Each of the two infinite intervals contains one real root of $Q(x)$.
Case 2. n is odd.
Regardless of whether $a > 0$ or $a < 0$, $P(x) < 0$ on $\frac{n-1}{2}$ finite intervals and one infinite intervals. The number of distinct real roots of $Q(x)$ is therefore $2(\frac{n-1}{2}) + 1 = n$.

11. Let p be a prime and let $0 < d < p$. Then p and d are relatively prime and there exists a positive integer k such that $kp+1$ is a multiple of d, say $kp+1 = md$ for some integer $m > 1$. Then $\frac{1}{d}$ is the sum of k copies of $\frac{p-d}{d}$ minus the sum of $(m-1)(p-1)$ copies of $\frac{1}{p-1}$. Now let n be a composite number. Let p be a prime divisor of n and let k be the positive integer such that $p^k < n \le p^{k+1}$. There are no fractions whose denominators are multiples of p^{k+1}. Suppose the denominator is ap^k where $0 < a < p$. Then the fraction is $\frac{n-ap^k}{ap^k}$, and at least one copy of p will be cancelled. It follows that there are no possible expressions for $\frac{1}{p^k}$.

12. We first show by an example that four red chameleons are not sufficient. Suppose that the rules are as follows. Any chameleon bitten by a blue one turns blue. Any red chameleon bitten by another red one turns yellow. Any yellow chameleon bitten by another yellow one turns white. Any white chameleon bitten by another white one turns black. Any black chameleon bitten by another black one turns blue. There are no changes in colors in all other cases. Starting with four red chameleons, we can get at most three yellow ones, two white ones, a black one, but not a blue one. Starting with five or more red chameleons, we can make all of them blue. We now prove that it is sufficient to start with five red chameleons by whatever rules which can make 2023 red chameleons all blue. We number the colors in order of their first appearances in the large group of 2023 chameleons. Thus red is color 1. The first chameleon of color 2, say yellow, must be red before and has been bitten by another red one. The first chameleon of color 3, say white, may be red or yellow before and bitten by another red or yellow one. If these two are the same chameleon, we can create another yellow chameleon as before. Analogously, we may have a first chameleon of color 4, say black, and a first blue chameleon which is color 5. At this critical moment, the large group consists of one chameleon of each color except that there are still 2019 red ones. Since the same rules apply to the small group of 5 chameleons, we can arrange for it to consist of one chameleon of each color.

We now renumber the colors in order of their last disappearance in the large group. Thus blue is still color 5 while red may no longer be color 1. For instant, suppose color 1 is black. Then the last black chameleon is bitten by a chameleon of another color, say yellow. Then in the small group, the yellow chameleon will bite the black one and eliminate color 1. When the chameleon of color 2 is eliminated from the large group, this will also happen in the small group. Since eventually all chameleons in the large group turn blue, so do all chameleons in the small group.

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