We mention just one.
\[
\frac{a}{s-a} + \frac{b}{s-b} + \frac{c}{s-c} = \frac{4R}{r} - 2 \geq 6
\]

\[\cos \alpha, \cos \beta, \cos \gamma \text{ satisfy} \]
\[4R^2x^3 - 4R(R + r)x^2 + (s^2 + r^2 - 4Rr)x - (2R + r^2) - s^2 = 0\]

\[\sigma_1: \cos \alpha + \cos \beta + \cos \gamma = 1 + \frac{3}{2} \leq \frac{r}{2} \text{ (just one of infinitely many examples)}\]

\[u = \sin^2 \frac{\alpha}{2}, \quad \frac{v}{2} = \sin^2 \frac{\beta}{2}, \quad w = \sin^2 \frac{\gamma}{2} \text{ satisfy} \]
\[16R^2x^3 - 8R(R + r)x^2 + (s^2 + r^2 - 8Rr)x - r^2 = 0\]

\[\sigma_2: u^2 + v^2 + w^2 = 1 \geq \frac{r^2}{4R} \geq \frac{r}{4}\]

\[\sqrt{\sigma_3}: \frac{u \cdot v \cdot w}{4R} \leq \frac{r}{8}\]

\[\alpha = \cos \frac{\alpha}{2}, \quad \frac{\beta}{2} = \cos \frac{\beta}{2}, \quad \frac{\gamma}{2} = \cos \frac{\gamma}{2} \text{ satisfy} \]
\[rx^3 - sx^2 + (4R + r)x - s = 0. \text{ This implies} \]
\[\sigma_2: uv + vw + uw + \frac{1}{uv} + \frac{1}{vw} + \frac{1}{uw} = 1\]

\[h_a, h_b, h_c \text{ satisfy} 2Rx^3 - (s^2 + r^2 + 4Rr)x^2 + 4Rr - 4s^2r^2 = 0. \text{ This implies} \]
\[\sigma_2: h_a h_b + h_a h_c + h_b h_c = \frac{2s^2r}{R h_a h_b h_c} = \frac{1}{R} + \frac{1}{h_a h_b} + \frac{1}{h_b h_c} + \frac{1}{h_a h_c} \]

Problems generated in this systematic way do not look particularly good to me. Most good problems are generalizations, specializations or small variations of known results. At least this is the way I produce most of the problems for the IMO or the German Mathematical Olympiad. A flash of inspiration strikes you only several times in your lifetime. So it is an unreliable source in problem creation.

**THE CREATION OF MATHEMATICAL OLYMPIAD PROBLEMS**

Arthur Engel

After graduating from the University of Stuttgart in 1952, Arthur Engel was a high school teacher for 18 years. He liked it so much in high school that he only reluctantly went to the Ludwigsburg College as an associate professor in 1970. Since 1972 he has been a full professor of didactics of mathematics at the mathematics department of the University of Frankfurt.

Since 1977 (its first participation) until 1984 he has been trainer and delegated leader of the IMO team of the Federal Republic of Germany. While giving up the leadership of the team in 1985 and 1986, he has continued to train the team.

It is far more difficult to create a problem than to solve it. There are very few routine methods of problem creation. As far as I know there was no Polyas among problem creators who wrote a book with the title *How To Create It*. By a problem I mean an Olympiad type nonroutine problem. Of course there are routine methods of creating routine problems.

One routine method of problem creation is based on the inversion of a

**Universal problem solving paradigm:**

You have a difficult problem. Transform it to make an easy problem. Invert it to get a solution to the difficult problem.

Problem creators often use the inverse procedure:

Start with an easy problem. Transform it to make it a difficult problem. Pose the problem to challenge problem solvers.

This is the:

**Poor man's problem creation method.**

**Example:**

How to create (discover, invent) a new triangular inequality.

Start with two universally known formulae for the triangle in fig. 1.

\[ab = \frac{1}{2} \sin \gamma, \quad c^2 = a^2 + b^2 - 2ab \cos \gamma\]
Solve for sine and cosine:

\[
\sin \gamma = \frac{2A}{ab}, \quad \cos \gamma = \frac{a^2 + b^2 - c^2}{2ab}
\]

There are innumerable relations between sine and cosine, such as

\[
\sin^2 \gamma + \cos^2 \gamma = 1, \quad \sin \gamma + \cos \gamma = \sqrt{2}, \quad \sin \alpha + \sin 2\alpha \leq \frac{3\sqrt{3}}{2}
\]

We have equality in (2) if and only if the triangle is equilateral.

Start with (2) and by a sequence of transformations wipe out all traces of its origins.

Expand:

\[
3(\sin \alpha + \sin \beta + \sin \gamma) + \cos \alpha + \cos \beta + \cos \gamma \leq 6
\]

Plug in (1) and multiply by 2abc:

\[
4A\sqrt{3}(a + b + c) + a(b^2 + c^2 - a^2) + b(a^2 + c^2 - b^2) + c(a^2 + b^2 - c^2) \leq 12abc
\]

This is still too easy. We must wipe out all traces of the cosine theorem

\[
4A\sqrt{3}(a + b + c) + a(b^2 + c^2 - a^2) + b(a^2 + c^2 - b^2) + c(a^2 + b^2 - c^2) \leq 12abc + 2a^3 + 2b^3 + 2c^3
\]

(3) \(a + b + c)(4A\sqrt{3} + a^2 + b^2 + c^2) \leq 12abc + 2a^3 + 2b^3 + 2c^3

We could stop here since (3) is by now a difficult problem. Or we could go on hiding traces of the path back to (2) by a further transformation using the identity

\[
a^3 + b^3 + c^3 = 3abc + (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)
\]

We get finally

\[
4A\sqrt{3} \leq \frac{18abc}{a + b + c} + a^2 + b^2 + c^2 - 2ab - 2bc - 2ca
\]

which is a very difficult problem to prove. Let us make the inequality simpler by making it weaker and thus irreversible. The harmonic mean is at most equal to the arithmetic mean.

So we get

\[
\frac{18abc}{a + b + c} = 6 \frac{3}{1/ab + 1/bc + 1/ca} \leq \frac{6 \frac{6}{3}}{ab + bc + ca} = \frac{2ab + 2bc + 2ca}{3} = 2ab + 2bc + 2ca
\]

We get

\[
4A\sqrt{3} \leq a^2 + b^2 + c^2
\]

This Weitzenbock Inequality (1919) is an Olympiad type problem and in fact it was used at the IMO III in 1961.

A sophisticated man's problem creation method

Sides a, b, c of a triangle satisfy the cubic equation

\[
x^3 - 2ax^2 + (b^2 + r^2 + 4Rr)x - 4srR = 0
\]

The elementary symmetric functions give

\[
\sigma_1 : a + b + c = 2s \quad (\text{trivial})
\]

\[
\sigma_2 : ab + bc + ca = s^2 + r^2 + 4Rr \quad (\text{hardly known})
\]

\[
\sigma_3 : abc = 4srR \quad (\text{well known})
\]

Every symmetric polynomial in a, b, c can be represented by \(\sigma_1, \sigma_2, \sigma_3\) and thus by \(s, r, R\).

We get infinitely many relations. For example

\[
a^2 + b^2 + c^2 = 2(s^2 - r^2 - 4Rr)
\]

\[
\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca} = \frac{1}{2R}
\]

If we set \(a = 2R\sin\alpha, b = 2R\sin\beta, c = 2R\sin\gamma\) in each of these relations, we get

\[
\sin \alpha + \sin \beta + \sin \gamma = \frac{s}{R}
\]

\[
\sin \alpha \sin \beta + \sin \beta \sin \gamma + \sin \gamma \sin \alpha = 4
\]

The last inequality results via the classic inequality \(R \geq 2r\), which follows from the fact that the circumference of the triangle of midpoints of a triangle is not smaller than the incircle.

Segments \(s - a, s - b, s - c\) of a triangle satisfy \(x^3 - sx^2 + sr(4R + r)x - sr^2 = 0\). Via the elementary symmetric functions this equation implies infinitely many relations.
Solve for sine and cosine:

\[
\begin{align*}
\sin \gamma &= \frac{2A}{ab}, \quad \cos \gamma = \frac{a^2 + b^2 - c^2}{2ab}.
\end{align*}
\]

There are innumerable relations between sine and cosine, such as

\[
\sin^2 \gamma + \cos^2 \gamma = 1, \quad \sin \gamma + \cos \gamma \leq \sqrt{2}, \quad \sin \alpha + \sin 2a \leq \frac{3\sqrt{3}}{4}.
\]

(2) \sin(\alpha + 30^\circ) + \sin(\beta + 30^\circ) + \sin(\gamma + 30^\circ) \leq 3

We have equality in (2) if and only if the triangle is equilateral. Start with (2) and by a sequence of transformations wipe out all traces of its origins.

Expand:

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3(\sin \alpha + \sin \beta + \sin \gamma) + \cos \alpha + \cos \beta + \cos \gamma \leq 6
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Plug in (1) and multiply by 2abc:

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\]

(3) \quad (a + b + c)(4A\sqrt{3} + a^2 + b^2 + c^2) \leq 12abc + 2a^3 + 2b^3 + 2c^3

We could stop here since (3) is by now a difficult problem. Or we could go on hiding traces of the path back to (2) by a further transformation using the identity

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So we get

\[
\frac{18abc}{a + b + c} = 6 \quad \frac{3}{1/ab + 1/bc + 1/ca} \leq 6 \quad \frac{ab + bc + ca}{3} = 2ab + 2bc + 2ca
\]

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Every symmetric polynomial in a, b, c can be represented by \(\sigma_1, \sigma_2, \sigma_3\) and thus by s, R, r. We get infinitely many relations. For example

\[
a^2 + b^2 + c^2 = 2(s^2 - r^2 - 4Rr)
\]

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\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca} = \frac{1}{2R}
\]

If we set a = 2Rsina, b = 2Rsin\beta, c = 2Rsin\gamma in each of these relations, we get

\[
\sin \alpha + \sin \beta + \sin \gamma = \frac{s}{r}
\]

\[
\frac{1}{\sin \alpha} + \frac{1}{\sin \beta} + \frac{1}{\sin \gamma} = \frac{2R}{s} \geq 4
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The creation of mathematical Olympiad problems

Arthur Engel

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Start with two universally known formulas for the triangle in fig. 1.

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A = \frac{1}{2} a b \sin \gamma, \quad c^2 = a^2 + b^2 - 2 a b \cos \gamma
\]
4 The Commission will make arrangements for the organization of a Pan-African Olympiad.

Any mathematician interested in the objectives and the Action Plan of this Commission are requested to write to the President:

Professor M. Akhtar,
Maison du Maroc,
1, boulevard Jourdan,
75014 - Paris,
FRANCE.

***

I will discuss in great detail the origin and variations of some problems I proposed to the IMO, which were finally accepted or at least were preselected by the host country.

London 1979

Define the bold gamble: My current fortune is \( x \leq 1 \), my goal is 1. If \( x = 1 \) then I quit.
If \( 0 < x \leq 0.5 \) then I stake \( x \) and have. If I win my fortune becomes \( 2x \), a loss brings me.
If \( 0.5 < x < 1 \) then I bet \( 1 - x \). If I win then I have the fortune \( x + (1 - x) = 1 \) and I quit.
If I lose I still have \( x - (1 - x) = 2x - 1 \) left.

Suppose in one round my chance of winning is \( p \) and my chance of losing is \( q \), \( p + q = 1 \).
Let \( f(x) \) be the probability of eventual success under bold play starting with fortune \( x \).
Then we have

\[
\begin{align*}
f(x) &= p f(2x), & 0 < x \leq 0.5 \\f(x) &= p + q f(2x - 1), & 0.5 < x < 1 \\f(0) &= 0, f(1) = 1
\end{align*}
\]

The above functional equation uniquely defines a function \( f \), which is very strange indeed, called singular. It is almost everywhere differentiable with \( f'(x) = 0 \), yet it is continuous and increasing.
For any rational \( x \) I can actually find \( f(x) \), although sometimes with considerable effort, depending on the period of the binary expansion of \( x \). "Closed" expression is out of the question.

My original idea was to ask for \( f(1/1979) \). If it had been the year 1984 it would have been a nice little problem, as 1984 = 29 * 31. So

\[
\begin{align*}
f(1/1984) &= p^2 f(1/31) \\
f(1/31) &= p^2 f(16/31) = p^2 + p^4 f(1/31) \\
f(1/31) &= p^2 f(1 - p^2) \\
f(1/1984) &= p^5 f(1 - p^4) \quad \text{with} \ f(1/1984) = 1/1984 \quad \text{for} \ p = 1/2.
\end{align*}
\]

But 1979 is a maximally nasty prime since 2^1979 = 1 (mod 1979) and no smaller exponent will produce 1. Thus the binary expansion of 1/1979 has 1979 an impossible task. So we have to look for a new question...

But wait, there is a probabilist on the jury. He might recognize the origin of the problem. So hide it.

(a) Specialize: \( p = 1/4, q = 3/4 \) (it is still a bold gamble)
(b) Switch \( p \) and \( q \) in the second equation (no more bold gamble)

Now we have

\[
\begin{align*}
f(x) &= \frac{1}{4} f(2x), & 0 < x \leq 0.5 \\
f(x) &= \frac{3}{4} + \frac{1}{4} f(2x - 1), & 0.5 < x < 1
\end{align*}
\]

but we have no good question. This functional equation still has to do with the binary expansion of \( x \). So try binary expansion. What happens to \( x = 0.b_1b_2b_3... \) in binary? It doubles up every digit.

Now we have our question

Let \( x \) be a rational with binary expansion \( x = 0.b_1b_2b_3... \)
Show that \( f(x) = 0.b_1b_2b_3b_4b_5... \).

1979 becomes free for the next problem. What is special about it? It is a prime. Make a problem which makes essential use of this property.
I knew two facts.

(a) \[
\frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{2n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{2n+1} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{2n} + \frac{1}{2n+1}
\]

This is well known and it has occurred in many competitions.

(b) \[
\frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{n+x} + \frac{1}{2n-x+2} + \frac{1}{2n} + \frac{1}{2n+1}
\]

1 + \frac{1}{n+x} + \frac{1}{2n-x+2} = \frac{3n+2}{(n+x)(2n-x+2)}

from the two ends has the same numerator 3n +2. This is well known. The motive has already been used by Gauss as a schoolboy.

Set 3n +2 = 1979. Then \( n = 599, n + 1 = 660, 2n + 1 = 1310 \). Now 1979 has disappeared.

Problem:

Let \( \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{p} + \frac{1}{q} = n \), with \( p, q \) from \( N \). Show that 1979 | \( p \).

Two easy ideas in series result in a very different problem.


Problem:

The average of all minima of all r-subsets of an n-set is \( f(n,r) = \frac{n+1}{r+1} \).

A simple result has a simple proof. After considerable effort I found a solution without computation, just by insight. It was a probabilistic solution and it runs as follows:-

Take \( n + 1 \) equally spaced points on a circle of length \( n + 1 \). Choose \( r + 1 \) of the \( n + 1 \) points at random. The chosen points split the circle into \( r + 1 \) parts. By symmetry each part has the same expected length \( (n+1)/(r+1) \). Cut the circle at the \( (r+1) \)th chosen point and straighten it into a segment of length \( n+1 \) (see fig. 2).

![Fig. 2](image)

Then I have \( r \) points chosen at random along the points 1, 2, \ldots, n, and we have

\[
E(\text{Minimum}) = \frac{n+1}{r+1}
\]
THE MATHEMATICAL ASSOCIATION OF CYPRUS AND SOME OF ITS ACTIVITIES

George Philippou

Dr. George Philippou is a Lecturer of Mathematics at the Higher Technical Institute, Nicosia. He has been the secretary of the Mathematical Association of Cyprus since it was established and he was also a deputy leader of the Cyprus delegation for the 26th IMO in Finland.

The Mathematical Association of Cyprus (MAC) was established in 1983. Some earlier efforts had only partial and temporary success and they would not last for long. This time almost 90% of the target population have already joined the MAC offering help towards the realization of its major objective i.e. "the promotion of mathematics and mathematics education."

From the first day of our organization we have set down to work and within the first two years we have been active in the following areas:

Two National Competitions are now organized on a yearly basis, one for the Gymnasia (K7-K9) and one for the Lyceums (K10-K12). The second one is called the Cyprus Mathematical Olympiad and its results form the basis for the selection of the National team which participates in international competitions. We have already participated in the 25th and 26th IMO's in Czechoslovakia and Finland, winning one bronze medal in each case. We have also participated in the Balkanian Olympiad which was held in Romania in May 1986, with the same success again, winning one bronze medal.

As from 1984 the MAC publishes its own magazine, Mathematica Bema, addressed to both students and teachers. For the time being, it is published once a year.

A number of lectures have also been organized on a yearly programmed plan, on special topics of mathematics and on the pedagogy of mathematics. Normally these lectures 'travel' to most towns of Cyprus.

The members of the governing committee are faced with a number of problems for the following reasons:

(i) the lack of experience, since we are still at the beginning of the journey,

(ii) economic resources which are really limited for one additional reason: the small size of the country and consequently the number of members, and

(iii) the non-existence of the University of Cyprus which would have played a real role in promoting our objectives from a different point of view.

Even so, we are optimistic for a better future and we value our relations with other national and international organizations very highly.

Dr. George Philippou,
Higher Technical Institute,
P. O. Box 2433,
Nicosia, CYPRUS.

***

I wanted to find out if there is a combinatorial solution without computation. So I proposed the problem at the IMO. There does not seem to exist a combinatorial solution of comparable simplicity. Only one Polish student produced a combinatorial solution but with very complicated reasoning.


Given relatively prime coins of denominations $a_1, a_2, \ldots, a_k$. Determine the largest amount which cannot be formed by means of these coins.

Let $g(a_1, a_2, \ldots, a_k)$ be the largest amount not payable. It is well known that

$$g(a_1, a_2) = a_1a_2 - a_1 - a_2 \quad \text{(Sylvester)}$$

For $k \geq 3$ the problem is unsolved. That is, for $(a,b,c) = 1$ the largest number, which cannot be represented by the form

$$ax + by + cz \quad \text{with nonnegative integers},$$

is not known.

Find a special case of the right order of difficulty, which is unknown or at least not known to the jury. Knowledge of Sylvester's result would not help much. Working for two months on the problem, off and on for 15 minutes, I produced the following problem:

Let $a, b, c = 1$. Then $2ab - ac - bc - ca$ is the largest number that cannot be represented in the form $bcx + cay + abz$ with nonnegative integers $x, y, z$.

First I thought that Sylvester's result does not help much, but I overlooked $bcx + cay + abz = c(bx + ay) + abz = cu + abz$, with $(a, b) = (c, ab) = 1$.

There are algorithms which quickly solve the Frobenius problem. That is, for any numerical input $a_1, a_2, \ldots, a_k$ with $(a_1, a_2, \ldots, a_k) = 1$, they produce the largest integer $G$ which cannot be represented in the form

$$a_1x_1 + \ldots + a_kx_k \quad \text{with nonnegative integers} \quad x_i.$$

For any smaller number they produce a solution if there is one. For any $L > G$ they always produce a solution of

$$a_1x_1 + \ldots + a_kx_k = L.$$

In this respect the Frobenius problem is solved. No 'closed' formula has ever been found. It may not exist or it may be so complicated as to be useless.

Washington 1981 (again)

For elementary transcendental functions there are deep and miraculous duplication formulae which allow the efficient computation of these functions. They are based on the properties of the circle $x^2 + y^2 = 1$ and the hyperbola $xy = 1$. Indeed, all these functions can be defined via the circle and the hyperbola. Does there exist a similar duplication formula for the parabola?

Let $A_n$ be the approximation of the area under the parabola $y = x^2$ from $x = 0$ and $x = 1$ by $n$ equally wide rectangles (Fig. 3). Then $A_n$ is the approximation of the same area by $2n$ rectangles.

![Fig. 3](image)
We look for a relation 
\[ A_{2n} = f(A_n) \] By eliminating \( 1/n \) from 
\[ A_n = \frac{1}{6} \left( \frac{1}{2} + \frac{1}{n} \right) \text{ and } A_{2n} = \frac{1}{6} \left( \frac{1}{2} + \frac{1}{n} \right) \]

we get 
\[ A_{2n} = \frac{1 + 4A_n + \sqrt{(1 + 24A_n)}}{16} \]

This led to the following problem:

Let \( a_1 = 1, a_{n+1} = \frac{1 + 4a_n + \sqrt{(1 + 24a_n)}}{16} \). Find a closed formula for \( a_n \).

Prague 1984. The van der Waerden Conjecture

Let \( S \) be an \( n \times n \) matrix with elements \( a_{ik} \). The permanent of \( S \) is defined by

\[ \text{per}(S) = \sum_{\sigma} a_{1(\sigma(1))} a_{2(\sigma(2))} \cdots a_{n(\sigma(n))} \]

The summation is to be extended over all permutations \( \sigma \) of \( \{1, 2, \ldots, n\} \). \( S \) is called doubly stochastic if its elements \( a_{ik} \) are nonnegative and the rows as well as the columns add up to 1.

Van der Waerden conjectured in 1927 that for doubly stochastic matrices

\[ \text{per}(S) \geq \frac{n!}{n^n} \text{ with equality iff } a_{ik} = \frac{1}{n} \text{ for all } i,k. \]

The conjecture was proved just before 1984. I decided to look at the case \( n = 3 \) in the hope that even this case is nontrivial. A recent book by H. Minc on permanents, which was written before the proof of the van der Waerden Conjecture showed that the case \( n = 3 \) was solved, but was indeed nontrivial.
"Clubs for Young Mathematicians" in the districts, "Mathematical Societies for Pupils" of some universities, in holiday camps etc.

In memory of C. F. GAUSS (1777-1855) a regular polygon with 17 vertices, a compass and a ruler are utilized as the symbol of the OJM.

The participants are divided, corresponding to their instruction, in "olympiad-classes" (OK1), which correspond in general with the form in school. In the 5th class are pupils aged 11 years, in the 12th class pupils aged 18 years. But the pupils of the 12th class also start in the OK1. Each pupil can start in a higher OK1 than his class in the school.

The contest is performed in 4 rounds:

1. The "Schul-OJM" occurs in September each year in the nearly 6,000 schools of the country. There has been no obligatory format to this competition since the 10th OJM. In some journals (for teachers or pupils, e.g. "alpha") 4 problems for each OK1 from 5 to 11 are published. In October, the solutions are published. The number of participants is not registered now. The maximum (nearly one million) was in the 6th OJM.

2. The "Kreis-OJM" occurs on a Wednesday in November. In the DDR there are nearly 230 districts named "Kreise" or "Stadtbezirke" in Berlin. The participants have to solve 4 problems each in OK1 from 5 to 11. They work on them for a maximum of 5 hours. The number of participants is nearly 50,000 in the whole country.

3. The "Bezirks-OJM" occurs on a Saturday/Sunday in February. In the DDR there are 15 districts named "Bezirke" of Berlin, capital of the DDR. The participants have to solve 3 problems each on the two days in OK1, 7 to 11. They work on them for a maximum of 4 hours. The number of participants is nearly 2,500 in the whole country.

4. The "DDR-OJM" takes place during 4 days in May, now at the "Pedagogical High School" in Erfurt. The nearly 200 participants have to solve 3 problems each from 2 in the OK1 10 and 11. They work on them each day for a maximum of 4 1/2 hours.

In order to continue on to the Bezirks-OJM and the DDR-OJM, students must have participated successfully in the Kreis-OJM. Participants are anonymous (the papers are numbered) and eligibility for progression is determined by a committee.

Problems and solutions for all tests are prepared by the "Central Committee for the OJM." Problems are based on school knowledge and are chosen from arithmetic and elementary number theory, equations, inequalities, functions (but without infinitesimal calculus as a rule), combinatorics, elementary geometry in the plane and in the space, geometrical constructions. In individual cases, problems from group theory, theory of probability, operations research etc. have been chosen. Teachers receive a copy of the solutions to the problems.

Students are limited in regard to the equipment they can use. A compass, ruler and slide rule are allowed, however a decision has not yet been taken regarding the use of calculators. In the Kreis-OJM and in the Bezirks-OJM, pupils can also use school collections of formulas and tables of functions.

The best participants get certificates and small prizes (e.g. books) in a publicized ceremony. The members for the DDR-team at the IMO are selected from the best participants of the DDR-OJM.

Further information regarding the organization, problems and the development of the participants in the DDR-OJM can be obtained from the following literature.


So I started with

\[
S = \begin{pmatrix}
    a_1 & a_2 & a_3 \\
    b_1 & b_2 & b_3 \\
    c_1 & c_2 & c_3
\end{pmatrix}, \quad 1 = (2a) (2b) (2c) = \text{per} (S) + 21 \text{ terms}
\]

and began to transform furiously until miraculously or by some error the following problem resulted.

\[
x, y, z \geq 0, x + y + z = 1 \implies 0 \leq xy + yz + zx - 2xyz \leq \frac{7}{27}
\]

Later I could not reconstruct the road leading to this inequality. Thorough check in Mitrinovic's "Elementary Inequalities" and sporadic check in his "Analytic Inequalities" showed that the left side of the inequality was known, but not the right side. So I sent the problem to Prague with the remark that it is a proof of van der Waerden's Conjecture for n = 5. It turned out to be too easy.

**Prague 1984**

At that time I was writing a book on statistics and computing. In statistics one is interested in questions of the type: Is \( p = 1453226798 \) a "random" permutation, or is there a trend upwards?

In \( p \) there are 36 pairs, of which only 6 are falling and 30 are rising. The 6 out of sort pairs are called inversions. By symmetry a random 9-permutation is expected to have 18 rising and 18 falling pairs.

What is the probability \( P \) that a random 9-permutation has 6 or even less inversions?

To answer questions of this type we must count

\[ p(n,k) = \text{number of } n\text{-permutations at most } k \text{ inversions} \]

I found a computer friendly recursion formula for \( p(n,k) \). It occurs neither in Kendall's "Rand Correlation Methods" nor in Knuth's Computer Science Bible. So I sent to Prague the following problem.

In a permutation \( (x_1, \ldots, x_n) \) of \( S = \{1,2,\ldots,n\} \) we call a pair an inversion if \( i < j \) and \( x_i > x_j \). Let \( \text{per}(n,k) \) be the number of permutations with at most \( k \) inversions.

- a) Find \( \text{per}(n,2) \).
- b) Show that \( \text{per}(n,k) = \text{per}(n,k-1) + \text{per}(n-1,k) - \text{per}(n-1,k-1) \) with \( \text{per}(n,0) = n! \) and \( \text{per}(n,1) = 1 \).

Compute with this recursion a table of \( p(n,k) \) for \( n \leq 9 \) and \( k \leq 6 \).

It turns out that \( p(9,6) = 2298 \) and \( P = 2298/9! = 0.0063 \). So \( p = 1453226798 \) is definitely not "random". In statistics we do not believe in miracles.

**German Mathematical Competition**

Here problems are much easier to find. The result can be well known, but should not be available in the German literature. On the other hand we sometimes require generalizations. Such open ended problems are even harder to create.
Let us take a typical example (a well known problem).

Look at Fig. 4. C₁ touches C₄⁺₁ in T₁ (C₃ = C₁). Show that X₁ = X₃.

Generalize!

The student is expected to find

(a) For three circles X₁ and X₄

are antipodes on C₁.

(b) Generally, for an even number

n of circles X₁ = X₄⁺ₙ₁, and for

odd n X₁ and X₄⁺₁ are antipodes

on C₁.

(c) Everyting depends on the parity

of the number of external points

tangency.

Bold generalization should reward the student by a lucid solution.

Example:

Start with two piles containing a and b checkers, respectively. Repeatedly double the number of checkers on the smaller pile at the expense of the larger pile. In the solution (c, c) move to (0, 2c) and stop.

Investigate this algorithm!

The student is expected to answer the following questions.

(a) When does the algorithm stop, and if so, in how many steps?

(b) When do we have a pure cycle?

(c) Find the minimal cycle length c depending on a and b.

(d) Investigate the case of real a and b.

If a student thinks of (d) he will be rewarded by a simple solution.

Let a+b = n.

If a < n/2 then a ← 2a

If a ≥ n/2 then a ← a - b

But a-b = a - (n-a) = 2a - n = 2a (mod n).

So our algorithm reduces to

while a < > 0 do a ← 2a (mod n)

That is, we double the first pile repeatedly modulo n.

Write a/n in the binary system.

1) \[ a = 0.b₁b₂...bₖ \] (binary rational)

n

We have 2ka = 0 (mod n) for the first time. The algorithm stops after exactly k steps.

THE OLYMPIADS OF YOUNG MATHEMATICIANS (OJM) IN THE GERMAN DEMOCRATIC REPUBLIC (DDR).

Wolfgang Engel

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the DDR.

In the year 1959 eight pupils out of schools in the DDR, who were selected on the basis of their high marks in the certificate of maturity, took part in the 1st International Mathematical Olympiad (IMO) in Roumania, without success. 1960 and 1961 mathematical competitions were organized, in which pupils from Berlin, capital of the DDR, and Leipzig participated, but the first Olympiad of Young Mathematicians (OJM) for the whole country took place in the scholastic year 1961/62.

From 1962/63 the Ministry of Education, the Mathematical Society of the German Democratic Republic and the Central Council of the Free German Youth - the organization of the Youth in the DDR - have arranged the OJM.

The Central Committee for the OJM is responsible for the management of the OJM. It is composed of nearly 10 members, who are selected by the Ministry of Education, the Mathematical Society of the DDR, the Central Council of the Free German Youth and the Ministry of Higher Education. In the districts there are also local committees. The organization for the contest is carried out by the authorities for public instruction. All expenses are paid by them.

The aims of the OJM are as follows:

- to contribute, so that the pupils acquire inside and outside of the instruction respectable knowledge and ability in the field of mathematics, that they broaden their knowledge and that the pupils are educated to mathematical thinking;

- to make clear to all pupils the increasing importance of mathematics;

- to awaken or deepen interest for mathematics in the majority of the pupils;

- and last but not least, to find pupils, who are interested and gifted in mathematics, in order to make possible their furtherance systematically.

As well, the problems afford an opportunity for the teachers in their development in mathematics.

The participation in the OJM is voluntary. The competitions culminate in the activity of the pupils in mathematics outside lessons, which results in "study-groups" in the schools, in
1983  Sixth Form Level, Final Round.

1. Assume that \(a,b,c > 0\). Prove or disprove the proposition that

\[
a \cos^2 \theta + b \sin^2 \theta < c
\]

implies

\[
\sqrt{a} \cos^2 \theta + \sqrt{b} \sin^2 \theta < \sqrt{c}.
\]

2. Show that no perfect square can end with 46 as its last two digits.

3. A duck, swimming near the middle of a circular pond, wishes to fly away but due to a damaged wing it can only take off from the bank. A fox is prowling round the edge of the pond trying to catch the duck without getting wet. Given that the fox can run four times as fast as the duck can swim, can the duck reach the bank and escape? Explain your answer.

4. Find two fractions having positive integral numerators and denominators 7 and 9 such that their sum is

\[
\frac{10}{63}.
\]

Explain your solution.

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1988 IMO

The 1988 IMO will be hosted by Australia as part of its bicentennial celebrations.

For further information please contact:-

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2) \[
\frac{a}{n} = 0.b_1b_2 \ldots b_n \overline{b_1b_2} \ldots \]

We have a tail of length \(t\) and a cycle of length \(c\), as in Fig. 5.

3) \[
\frac{a}{n} = 0.b_1b_2 \ldots \text{ non-periodic and non-terminating}.
\]

No stopping, no periodicity.

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