THE CREATION OF MATHEMATICAL OLYMPIAD PROBLEMS

Arthur Engel

After graduating from the University of Stuttgart in 1952, Arthur Engel was a high school teacher for 18 years. He liked it so much in high school that he only reluctantly went to the Ludwigsburg College as an associate professor in 1970. Since 1972 he has been a full professor of didactics of mathematics at the mathematics department of the University of Frankfurt.

Since 1977 (its first participation) until 1984 he has been trainer and delegation leader of the IMO team of the Federal Republic of Germany. While giving up the leadership of the team in 1985 and 1986, he has continued to train the team.

It is far more difficult to create a problem than to solve it. There are very few routine methods of problem creation. As far as I know there was no Polya among problem creators who wrote a book with the title ***How To Create It.***

By a problem I mean an Olympiad type nonroutine problem. Of course there are routine methods of creating routine problems.

One routine method of problem creation is based on the inversion of a

Universal problem solving paradigm:

You have a difficult problem. Transform it to make an easy problem. Invert it to get a solution to the difficult problem.

Problem creators often use the inverse procedure:

Start with an easy problem. Transform it to make it a difficult problem. Pose the problem to challenge problem solvers.

This is the:-

Poor man's problem creation method.

Example:

How to create (discover, invent) a new triangular inequality.

Start with two universally known formulae for the triangle in fig. 1.

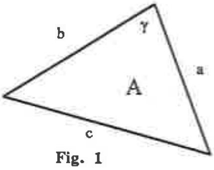

Fig. 1

$$A = \frac{ab}{2}\sin\gamma, \quad c^2 = a^2 + b^2 - 2ab\cos\gamma$$

Solve for sine and cosine:

$$(1)\quad \sin\gamma = \frac{2A}{ab}, \qquad \cos\gamma = \frac{a^2 + b^2 - c^2}{2ab}$$

There are innumerable relations between sine and cosine, such as

$$\sin^2\gamma + \cos^2\gamma = 1, \quad \sin\gamma + \cos\gamma \le \sqrt{2}, \quad \sin\alpha + \frac{1}{2}\sin 2a \le \frac{3\sqrt{3}}{4},$$

$$(2)\quad \sin(\alpha + 30^o) + \sin(\beta + 30^o) + \sin(\gamma + 30^o) \le 3$$

We have equality in (2) if and only if the triangle is equilateral.
Start with (2) and by a sequence of transformations wipe out all traces of its origins.
Expand:

$$3(\sin\alpha + \sin\beta + \sin\gamma) + \cos\alpha + \cos\beta + \cos\gamma \le 6$$

Plug in (1) and mutiply by 2abc:

$$4A\sqrt{3}(a + b + c) + a(b^2 + c^2 - a^2) + b(a^2 + c^2 - b^2) + c(a^2 + b^2 - c^2) \le 12abc$$

This is still too easy. We must wipe out all traces of the cosine theorem

$$4A\sqrt{3}(a + b + c) + a(a^2 + b^2 + c^2) + b(a^2 + b^2 + c^2) + c(a^2 + b^2 + c^2) \le 12abc + 2a^3 + 2b^3 + 2c^3$$

$$(3)\quad (a + b + c)(4A\sqrt{3} + a^2 + b^2 + c^2) \le 12abc + 2a^3 + 2b^3 + 2c^3$$

We could stop here since (3) is by now a difficult problem. Or we could go on hiding traces of the path back to (2) by a further transformation using the identity

$$a^3 + b^3 + c^3 = 3abc + (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$$

We get finally

$$(4)\quad 4A\sqrt{3} \le \frac{18abc}{a + b + c} + a^2 + b^2 + c^2 - 2ab - 2bc - 2ca$$

which is a very difficult problem to prove. Let us make the inequality simpler by making it weaker and thus irreversible. The harmonic mean is at most equal to the arithmetic mean.

So we get

$$\frac{18abc}{a+b+c} = 6\frac{3}{1/ab + 1/bc + 1/ca} \leq 6\frac{ab+bc+ca}{3} = 2ab + 2bc + 2ca$$

We get

$$4A\sqrt{3} \leq a^2 + b^2 + c^2$$

This Weitzenbock Inequality (1919) is an Olympiad type problem and in fact it was used at the IMO III in 1961.

A sophisticated man's problem creation method

Sides a, b, c of a triangle satisfy the cubic equation

$$x^3 - 2sx^2 + (s^2 + r^2 + 4Rr)x - 4sRr = 0$$

The elementary symmetric functions give

σ_1 : $a + b + c = 2s$ (trivial)

σ_2 : $ab + bc + ca = s^2 + r^2 + 4Rr$ (hardly known)

σ_3 : $abc = 4sRr$ (well known)

Every symmetric polynomial in a, b, c can be represented by $\sigma_1, \sigma_2, \sigma_3$ and thus by s, R, r. We get infinitely many relations. For example

$$a^2 + b^2 + c^2 = 2(s^2 - r^2 - 4Rr)$$

$$(a + b)(b + c)(c + a) = 2s(s^2 + r^2 + 2Rr)$$

$$\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca} = \frac{1}{2Rr}$$

If we set $a = 2R\sin\alpha$, $b = 2R\sin\beta$, $c = 2R\sin\gamma$ in each of these relations, we get

$$\sin\alpha + \sin\beta + \sin\gamma = \frac{s}{r}$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

$$\frac{1}{\sin\alpha\sin\beta} + \frac{1}{\sin\beta\sin\gamma} + \frac{1}{\sin\gamma\sin\alpha} = \frac{2R}{r} \geq 4$$

The last inequality results via the classic inequality $R \geq 2r$, which follows from the fact that the circumference of the triangle of midpoints of a triangle is not smaller than the incircle.

Segments $s - a$, $s - b$, $s - c$ of a triangle satisfy $x^3 - sx^2 + r(4R + r)x - sr^2 = 0$. Via the elementary symmetric functions this equation implies infinitely many relations.

We mention just one.

$$\frac{a}{s-a} + \frac{b}{s-b} + \frac{c}{s-c} = \frac{4R}{r} - 2 \geq 6$$

$\cos\alpha, \cos\beta, \cos\gamma$ satisfy

$$4R^2x^3 - 4R(R+r)x^2 + (s^2+r^2-4r^2)x + (2R+r)^2 - s^2 = 0$$

$$\sigma_1 : \cos\alpha + \cos\beta + \cos\gamma = 1 + \frac{r}{R} \leq \frac{3}{2}$$ (just one of infinitely many examples)

$u = \sin^2\dfrac{\alpha}{2}$, $v = \sin^2\dfrac{\beta}{2}$, $w = \sin^2\dfrac{\gamma}{2}$ satisfy

$$16R^2x^3 - 8R(2R-r)x^2 + (s^2+r^2-8Rr)x - r^2 = 0$$

$$\sigma_1 : u^2 + v^2 + w^2 = 1 - \frac{r}{2R} \geq \frac{3}{4}$$

$$\sqrt{\sigma_3} : u.v.w = \frac{r}{4R} \leq \frac{1}{8}$$

$u = \cot\dfrac{\alpha}{2}$, $v = \cot\dfrac{\beta}{2}$, $w = \cot\dfrac{\gamma}{2}$ satisfy $rx^3 - sx^2 + (4R+r)x - s = 0$. This implies

$$\sigma_2 ; uv + vw + wu + \frac{4R}{r} + 1 \geq 9, \quad \frac{1}{uv} + \frac{1}{vw} + \frac{1}{wu} = 1$$

h_a, h_b, h_c satisfy $2Rx^3 - (s^2+r^2+4Rr)x^2 + 4s^2rx - 4s^2r^2 = 0$. this implies

$$\sigma_2 : h_ah_b + h_bh_c + h_ch_a = \frac{2s^2r}{R}, \quad \frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c} = \frac{1}{r}$$

Problems generated in this systematic way do not look particularly good to me. Most good problems are generalizations, specializations or small variations of known results. At least this is the way I produce most of the problems for the IMO or the German Mathematical Competition. A flash of inspiration strikes you only several times in your lifetime. So it is an unreliable source in problem creation.

I will discuss in great detail the origin and variations of some problems I proposed to the IMO, which were finally accepted or at least were preselected by the host country.

London 1979

Define the bold gamble : My current fortune is $x \le 1$, my goal is 1. If $x = 1$ then I quit. If $0 < x \le 0.5$ then I stake x, all I have. If I win my fortune becomes 2x, a loss ruins me. If $0.5 < x < 1$ then I bet $1 - x$. If I win then I have the fortune $x + (1 - x) = 1$ and I quit.. If I lose I still have $x - (1 - x) = 2x - 1$ left.
Suppose in one round my chance of winning is p and my chance of losing is q, $p + q = 1$.
Let f(x) be the probability of eventual success under bold play starting with fortune x.
Then we have

$$f(x) = p.f(2x), \qquad 0 < x \le 0.5$$
$$f(x) = p + q.f(2x - 1) \qquad 0.5 < x < 1$$
$$f(0) = 0, f(1) = 1$$

The above functional equation uniquely defines a function f, which is very strange indeed, called singular. It is almost everywhere differentiable with $f'(x) = 0$, yet it is continuous and increasing. For any rational x I can actually find f(x), although sometimes with considerable effort, depending on the period of the binary expansion of x. "Closed" expression is out of the question.
My original idea was to ask for f(1/1979). If it had been the year 1984 it would have been a nice little problem, as $1984 = 2^6.31$. So

$$f(1/1984) = p^6 f(1/31)$$
$$f(1/31) = p^4 f(16/31) = p^5 + p^4 q f(1/31)$$
$$f(1/31) = p^5/(1-p^4 q)$$
$$f(1/1984) = p^{11}/(1-p^4 q) \quad \text{with } f(1/1984) = 1/1984 \text{ for } p = 1/2.$$

But 1979 is a maximally nasty prime since $2^{1978} \equiv 1 \pmod{1979}$ and no smaller exponent will produce 1. Thus the binary expansion of 1/1979 has period 1978, an impossible task. So we have to look for a new question...
But wait, there is a probabilist on the jury. He might recognize the origin of the problem. So hide it.

(a) Specialize : $p = 1/4, q = 3/4$ (it is still a bold gamble)
(b) Switch p and q in the second equation (no more bold gamble)

Now we have

$$f(x) = \frac{1}{4} f(2x), \qquad 0 < x \le 0.5$$

$$f(x) = \frac{3}{4} + \frac{1}{4} f(2x - 1), \qquad 0.5 < x < 1$$

but we have no good question. This functional equation still has to do with the binary expansion of x. So try binary expansion. What happens to $x = 0.b_1b_2b_3...$ in binary? It doubles up every digit!
Now we have our question

Let x be a rational with binary expansion $x = 0.b_1b_2b_3...$
Show that $f(x) = 0.b_1b_1b_2b_2b_3b_3...$.

1979 becomes free for the next problem. What is special about it? It is a prime.
Make a problem which makes essential use of this property.

I knew two facts.

$$(a)\ 1 - \frac{1}{2} + \frac{1}{3} - \dots - \frac{1}{2n} + \frac{1}{2n+1} = 1 + \frac{1}{2} + \frac{1}{3} \dots + \frac{1}{2n+1} - 2\left(\frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2n}\right)$$

$$= \frac{1}{n+1} + \dots + \frac{1}{2n+1}$$

This is well known and it has occurred in many competitions.

$$(b)\ \frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{n+x} + \dots + \frac{1}{2n-x+2} \dots + \frac{1}{2n} + \frac{1}{2n+1}$$

$\dfrac{1}{n+x} + \dfrac{1}{2n-x+2} = \dfrac{3n+2}{(n+x)(2n-x+2)}$: The sum of the two terms the same distance from the two ends has the same numerator 3n +2. This is well known. The motive has already been used by Gauss as a schoolboy.

Set 3n + 2 = 1979. Then n = 959, n + 1 = 660, 2n + 1 = 1319. Now 1979 has disappeared.

Problem:

Let $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots + \frac{1}{1319} = \frac{p}{q}$ with p, q from N. Show that $1979 \mid p$.

Two easy ideas in series result in a very different problem.

Washington 1981.

Problem :

The average of all minima of all r-subsets of an n-set is $f(n,r) = \dfrac{n+1}{r+1}$

A simple result has a simple proof. After considerable effort I found a solution without computation, just by insight. It was a probabilistic solution and it runs as follows:-
Take n + 1 equally spaced points on a circle of length n + 1. Choose r + 1 of the n + 1 points at random. The chosen points split the circle into r + 1 parts. By symmetry each part has the same expected length (n +1)/(r +1). Cut the circle at the (r + 1)th chosen point and straighten it into a segment of length n + 1 (see fig. 2)

Fig. 2

Then I have r points chosen at random along the points 1, 2, . . . , n, and we have

$$E(\text{Minimum}) = \frac{n+1}{r+1}$$

I wanted to find out if there is a combinatorial solution without computation. So I proposed the problem at the IMO. There does not seem to exist a combinatorial solution of comparable simplicity. Only one Polish student produced a combinatorial solution but with very complicated reasoning.

Paris 1983 : The Frobenius Problem.

Given relatively prime coins of denominations $a_1, a_2, \ldots, a_k$. Determine the largest amount which cannot be formed by means of these coins.

Let $g(a_1, a_2, \ldots, a_k)$ be the largest amount not payable. It is well known that

$$g(a_1, a_2) = a_1a_2 - a_1 - a_2 \qquad \text{(Sylvester)}$$

For $k \geq 3$ the problem is unsolved. That is, for $(a,b,c) = 1$ the largest number, which cannot be represented by the form

$$ax + by + cz \quad \text{with nonnegative integers,}$$

is not known.
Find a special case of the right order of difficulty, which is unknown or at least not known to the jury. Knowledge of Sylvester's result would not help much. Working for two months on the problem, off and on for 15 minutes, I produced the following problem :

Let $(a,b) = (b,c) = (c,a) = 1$. Then $2abc - ab - bc - ca$ is the largest number that cannot be represented in the form $bcx + cay + abz$ with nonnegative integers x,y,z.

First I thought that Sylvester's result does not help much, but I overlooked

$$bcx + cay + abz = c(bx + ay) + abz = cu + abz, \quad \text{with } (a,b) = (c,ab) = 1$$

There are algorithms which quickly solve the Frobenius problem. That is, for any numerical input
$a_1, a_2, \ldots, a_k$ with $(a_1, a_2, \ldots, a_k) = 1$, they produce the largest integer G which cannot be represented in the form

$$a_1x_1 + \ldots + a_kx_k \quad \text{with nonnegative integers } x_i.$$

For any smaller number they produce a solution if there is one. For any $L > G$ they always produce a solution of

$$a_1x_1 + \ldots + a_kx_k = L.$$

In this respect the Frobenius problem is solved. No 'closed' formula has ever been found. It may not exist or it may be so complicated as to be useless.

Washington 1981 (again)

For elementary transcendental functions there are deep and miraculous duplication formulae which allow the efficient computation of these functions. They are based on the properties of the circle $x^2 + y^2 = 1$ and the hyperbola $xy = 1$. Indeed, all these functions can be defined via the circle and the hyperbola. Does there exist a similar duplication formula for the parabola ?

Let A_n be the approximation of the area under the parabola $y = x^2$ from $x = 0$ and $x = 1$ by n equally wide rectangles (fig. 3). Then A_{2n} is the approximation of the same area by 2n rectangles.

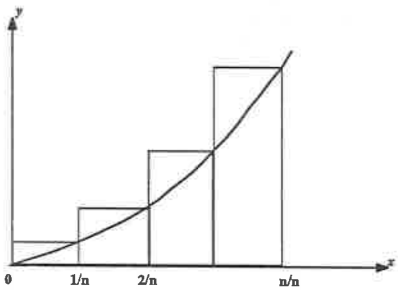

Fig. 3

We look for a relation
$A_{2n} = f(A_n)$. By eliminating 1/n from

$$A_n = \frac{1}{6}\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{2}\right) \text{ and } A_{2n} = \frac{1}{6}\left(1 + \frac{1}{2n}\right)\left(2 + \frac{1}{2n}\right)$$

we get

$$A_{2n} = \frac{1 + 4A_n + \sqrt{(1 + 24A_n)}}{16}$$

This led to the following problem:

Let $a_1 = 1$, $a_{n+1} = \dfrac{1 + 4a_n + \sqrt{(1 + 24a_n)}}{16}$. Find a closed formula for a_n.

Prague 1984. The van der Waerden Conjecture

Let S be an nxn matrix with elements a_{ik}. The permanent of S is defined by

$$\text{per}(S) = \sum_{\sigma} a_{1\sigma(1)}\, a_{2\sigma(2)} \cdots a_{n\sigma(n)}$$

The summation is to be extended over all permutations σ of {1, 2, ..., n}. S is called doubly stochastic it its elements a_{ik} are nonnegative and the rows as well as the columns add up to 1.

Van der Waerden conjectured in 1927 that for doubly stochastic matrices

$$\text{per}(S) \geq \frac{n!}{n^n} \text{ with equality iff } a_{ik} = \frac{1}{n} \text{ for all i,k.}$$

The conjecture was proved just before 1984. I decided to look at the case n = 3 in the hope that even this case is nontrivial. A recent book by H. Minc on permanents, which was written before the proof of the van der Waerden Conjecture showed that the case n = 3 was solved, but was indeed nontrivial.

So I started with

$$S = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}, \quad 1 = (\Sigma a)\,(\Sigma b)\,(\Sigma c) \; = \text{per}\,(S) + 21 \text{ terms}$$

and began to transform furiously until miraculously or by some error the following problem resulted.

$$x, y, z \geq 0, x+y+z = 1 \quad \Rightarrow \quad 0 \leq xy + yz + zx - 2xyz \leq \frac{7}{27}$$

Later I could not reconstruct the road leading to this inequality.
Thorough check in Mitrinovic's "Elementary Inequalities" and sporadic check in his "Analytic Inequalities" showed that the left side of the inequality was known, but not the right side. So I sent the problem to Prague with the remark that it is a proof of van der Waerden's Conjecture for $n = 3$. It turned out to be too easy.

Prague 1984

At that time I was writing a book on statistics and computing. In statistics one is interested in questions of the type: Is p = 1453226798 a "random" permutation, or is there a trend upwards?
In p there are 36 pairs, of which only 6 are falling and 30 are rising. The 6 out of sort pairs are called inversions. By symmetry a random 9-permutation is expected to have 18 rising and 18 falling pairs.
What is the probability P that a random 9-permutation has 6 or even less inversions?
To answer questions of this type we must count
p (n,k) = number of n-permutations at most k inversions
I found a computer friendly recursion formula for p (n,k). It occurs neither in Kendall's "Rand Correlation Methods" nor in Knuth's Computer Science Bible. So I sent to Prague the following problem.

In a permutation $(x_1, \ldots, x_n)$ of $S = \{1, 2, \ldots, n\}$ we call a pair an inversion if $i < j$ and $x_i > x_j$. Let p(n,k) be the number of permutations with at most k inversions.
a) Find p(n, 2).
b) Show that $p(n,k) = p(n,k-1) + p(n-1,k) - p(n-1, k-n)$
with $p(n,k) = 0$ for $k < 0$ and $p(n,0) = p(1,k) = 1$
Compute with this recursion a table of p(n,k) for $n \leq 9$ and $k \leq 6$.

It turns out that p(9, 6) = 2298 and P = 2298/9! = 0.0063. So p = 145326798 is definitely not "random". In statistics we do not believe in miracles.

German Mathematical Competition

Here problems are much easier to find. The result can be well known, but should not be available in the German literature. On the other hand we sometimes require generalizations. Such open ended problems are even harder to create.

Let us take a typical example (a well known problem).
Look at Fig. 4 . C_i touches C_{i+1} in T_i ($C_5 = C_1$). Show that $X_1 = X_5$. Generalize!

The student is expected to find

a) For three circles X_1 and X_4 are antipodes on C_1.

b) Generally, for an even number n of circles $X_1 = X_{n+1}$, and for odd n X_1 and X_{n+1} are antipodes on C_1.

c) Everying depends on the parity of the number of external points of tangency.

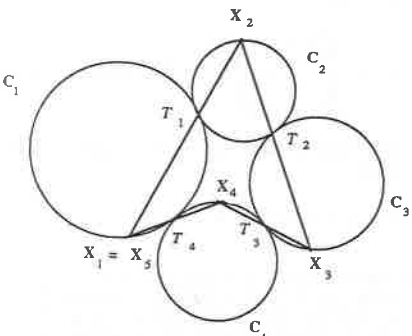

Fig. 4

Bold generalization should reward the student by a lucid solution.

Example:

Start with two piles containing a and b checkers, respectively. Repeatedly double the number of checkers on the smaller pile at the expense of the larger pile. In the positon (c, c) move to (0, 2c) and stop.
Investigate this algorithm!

The student is expected to answer the following questions.

(a) When does the algorithm stop, and if so, in how many steps?
(b) When do we have a pure cycle?
(c) Find the tail t and cycle length c depending on a and b.
(d) Investigate the case of real a and b.

If a student thinks of (d) he will be rewarded by a simple solution.
Let $a+b = n$.

If $a < n/2$ then $a \leftarrow 2a$

If $a \geq n/2$ then $a \leftarrow a - b$

But $a - b = a - (n - a) = 2a - n \equiv 2a \pmod n$.
So our algorithm reduces to

while a <> 0 do $a \leftarrow 2a \pmod n$

That is, we double the first pile repeatedly modulo n.

Write a/n in the binary system.

1) $$\frac{a}{n} = 0.b_1b_2 \ldots b_k \quad \text{(binary rational)}$$

We have $2^k a \equiv 0 \pmod n$ for the first time. The algorithm stops after exactly k steps.

2)

$$\frac{a}{n} = 0.b_1b_2 \ldots b_t\, d_1d_2 \ldots d_c$$

We have a tail of length t and a cycle of length c, as in Fig. 5.

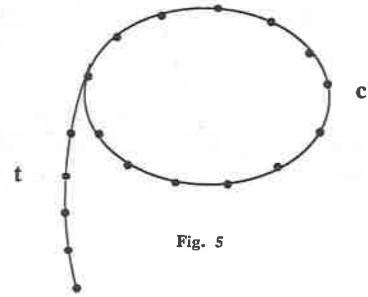

Fig. 5

3)

$$\frac{a}{n} = 0.b_1b_2b_3 \ldots \text{ nonperiodic and nonterminating.}$$

No stopping, no periodicity.

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