A classification of challenges

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This essay is an attempt to look at specific examples of challenge and try to clasify them. This may be of some use in helping us come to grips with describing what challenges are and how they might be used in various situation. The examples are mainly presented in terms of a classroom situation with a range of student ability represented.
However, the types have their analogues in challenges presented to particular groups and even to the general public.

I. Challenges in the presentation of mathematics

One of the roles of the teacher is to present mathematics that is new to the student and may be difficult to understand. Even when the teaching is competent, the background of the students is sound and the situation is appropriate, it may still be difficult for the student to grasp the point and understand the situation. Inevitably, a certain amount of time must pass and thinking occur before one is rewarded with comprehension. In this situation, there is a two-fold challenge:
for the teacher to present the material in the clearest possible way, and for the student to take intellectual ownership of the material.

This type of challenge recognizes, first that mathematics is often inherently difficult, and secondly that with some effort by those concerned, these difficulties might be surmounted. The advice given by Beatrice to the Pilgrim in the third book of Dante's "Divine Comedy" puts the matter quite well:

> you must sit at the table yet awhile
> because the food that you have taken in
> is tough and takes time to assimilate.
>
> Open your mind to what I shall reveal
> and seal it in, for to have understood
> and not retained, as knowledge does not count.
>
> Paradise, Canto V, 37-42

I do not include here material that is in the mainstream curriculum, although it may present it challenges to some students, because this is all in the ordinary course of business. But I do want to mention some things that deliberately stretch students, either by being quite deep or by providing the sort of subject matter, reasoning or process that is outside the syllabus.
Here are some examples:

(i) Epsilonics: A notoriously difficult topic for students to follow in a beginning analysis course is the formal definition of limit and continuity using epsilon and delta. My experience is that even with a class of reasonably well-prepared students, perhaps 5 to 10 per cent of the students grasp it before too long, and maybe another 20 to 30 per cent feel somewhat comfortable with it by the end of the course. In the final examinations, many students can go through the motions, but the marker feels that the technique is still poorly understood. However, by the time student get into a second or third course, they seem to have become comfortable with this level of formalism. There seems to be an inevitable process of mathematical maturation that can only occur over time.

It is interesting to speculate where the challenge of mastering this might be.

(a) Students often lack the algebraic skills to negotiating the mechanics of an epsilon-delta argument, particularly the handling of inequalities.

(b) The logic is rather subtle and many students have trouble sorting it out.

(c) Students may not see how the formal definition of limit or continuity accords with their intuitive understanding of these concepts. The formalism is static, while the intuition is dynamic -- one variable is changing in response to another. Intuitively, the independent variable is in the driver's seat and the dependent variable reacts to it. Formally, one is asked to set a degree of error for the dependent variable and then discover within what bounds the dependent variable can change to keep the dependent variable within the error.

(d) There is another way in which the formal definition may not appear to deliver what intuition seems to dictate. Students have a tendency to think about limits and continuity sequentially -- as a sequence of points tends to a limiting value, the corresponding functional values at these points tends to a particular value. On the face of it, how "fast" the functional values approach the limiting value might depend on the particular sequence chosen. However, according to the definition, given a positive epsilon, the dependent values are within epsilon of the limit whenever any sequence of independent values gets within epsilon of the value at which the limit is being taken.

(e) Applying the epsilon-delta process in proofs requires the making of coarse estimates, in which one can "throw away" a lot. Students at this point are used to solving equations for particular values, and are not initially able to appreciate making rough estimates that may not be very close, but close enough for one's pusposes. Making such estimates often requires a degree of judgment designed to save a lot of tedious labout.

(ii) Noughts and crosses (tic-tac-toe). A topic that can be taken up with students from the middle school on is the idea of a strategy that a player might adopt in a game, like noughts and crosses. This seems to be very difficult for some students to get hold of - the idea that one player can force either a win or a draw regardless of what steps the opponent might take (in distinction to happening to win or draw, when the right conditions obtain). A deeper level of difficulty occurs in the proof that the second player in the game of noughts and crosses cannot have a winning strategy (so that, if both players adopt their best strategies, either the first player must win or there must be a draw). The difficulty is compounded by the fact that the proof is by a contradiction argument. It is highly unlikely that most secondary students, indeed most university students, would come up with this particular argument. In fact, it is know that both players can force the game to a draw by best play; the establishment of this is straightforward, just a matter of following up the possibilities and making judicious use of symmetry. So the challenge here is not the actual result, but the nature of the reasoning employed.

Let A be the first player and B be the second player.
Suppose if possible, that the second player B has a winning strategy.
Then B can contrive to win regardless of how the A plays, in particular whether A begins with an X in the central cell, a corner cell, or a mid-edge cell.
A fortiori, B can force a win if A
takes the central cell. Let us call the strategy that achieves this, S.

Consider a new game, G, whose play is that of noughts and crosses, except that the central cell is a forbidden position and is automatically assigned to the second player. Let B be the first player in G and let B adopt the strategy S, which does not require using the central square. A, now as second player, cannot force a win or even a draw, because, if he could, he could apply the strategy to win the original game in which he plays first and additionally has the central cell in hand. So S is a winning strategy for the first player in the game G.

Return to noughts and crosses. Let B be the first player and adopt the strategy S. If the second player, now A, avoids the central cell, then B wins as he would have done as first player in G. Suppose, on the other hand, that the second player A grabs the central square. Then A will in effect have wasted a move, since B's strategy S does not require access to this square. Thus, B as first player would have a winning strategy for noughts and crosses. But this contradicts our assumption that the winning strategy belongs to the second player.

(iii) The marriage theorem. This result was actually presented to a Grade 10 "applied" class, the lower stream; as I had the time to be quite leisurely, the students seemed to follow it. In this theorem, we suppose that we have an equal number of men and women who have to be matched up into husband and wife couples.
Each person makes a list in strict order of preference of the members of the opposite sex. The theorem states that no matter how this is done, it is possible to pair off the men and women in such a way that no man prefers a woman to his own wife while at the same time that woman prefers him to her own husband.

The proof is algorithmic and proceeds in a number of rounds. In the first round, each man proposes to the top woman on his list; any woman receiving more than one proposal accepts provisionally only the most preferred among the suitors. If each man proposes to a different woman, then the process stops and we have achieved the result. Otherwise, there are some men who have been rejected, and we move to the second round. Each of these rejected men crosses his first choice off his list and proposes to his next choice. Each woman getting a proposal accepts the best among the provisional suitor (if any) from the first round and those proposing to her on the second round, and rejects the rest. This continues as long as there are rejected suitors.

Since at least one person gets crossed off someone's list whenever there is a rejection, the process must terminate after a finite number of rounds. Suppose, at the end of it, there is a man M who prefers some other wife W to his own. Then he must have proposed to W before proposing to his own wife and been rejected in favour of someone W preferred more; the husband that W ended up with would have been even higher on W's list of preferences.

This proof, in its dependence on pure verbal reasoning, is not what most students would regard as mathematics, and its challenge comes from its unfamiliarity.

(iv) An Olympiad problem. Here is a problem from an old IMO paper that I have taken classes through. It is unreasonable to expect a typical high school student to solve it, but many of them can follow and appreciate the underlying strategy. Think of it as a sonata in two movements.

Let A = 4444^(4444) (i.e. 4444 to the power 4444), B be the sum of the digits of A, C the sum of the digits to B and D the sum of the digits of C. What is D?

Movement 1: Estimate the sizes of A, B, C, D by obtaining an upper bound on the number of digits. There are a number of ways students can estimate the number of digits in A; the hard part is for them to understand that we are not looking for an exact answer.

One way is to note that $4444^{(4444)} < 10^{(4 \times 4444)} < 10^{(20000)}$, so A has no more than 20000 digits and the sum of these digits cannot exceed $9 \times 20000 = 180000$. Thus $B < 180000$, so $C < 1 + (5 \times 9) = 46$ and $D \leq 13$.

Movement 2: Cast out 9s. One determines that A is congruent to 7 (mod 9), as are B, C and D. Thus, we are looking for a number not exceeding 13 which leaves remainder 7 when divided by 9.

I have gone into some detail on these examples, as the idea of taking students through some sophisticated bit of mathematics is not as current as it possibly might be. Another example of a topic that takes some time to sink in is induction; some students get hung up on the prospect of assuming what they have to proof, and this appears to present a block. While the individual steps might be accessible to the students, there is quite a challenge when the whole thing is put together, as we are requiring the students to appreciate a strategy, follow a line of reasoning and perhaps engage in a type of mathematical discourse with which they are unfamiliar.

The challenge for the teacher and the education system is to present such things in a non-threatening way, in an atmosphere of exploration and discovery. (A visitor to a class, such as myself, might fare better than a regular teacher, because the students understand that they will not be held responsible for any transaction that might occur with them.)

II. Challenges provided for the student to work on

(a) Expository challenges: These arise in situations where students might naturally accept a fact to be true, but are suddenly forced to reflect on why it is true.

(v) There is a standard problem in which one has a square ABCD, and draws segments AP, BQ, CR, DS with P, Q, R, S the respective midpoints of BC, CD, DA, AB. One is asked what fraction of the area of the square is the area of the region bounded by AP, BQ, CR, DS. In a discussion with a grade 10 (age 15) class, everyone (including me) took it for granted that the interior figure was indeed a square, and it was only when some student asked why this was so that we began to look seriously at this question. There was a lot of consternation and students began to construct some pretty complicated arguments. Eventually, I got the students to realize that the symmetry of the figure was at the core of the matter, and it was a matter of describing what sort of symmetry it was. A straightforward argument is possible by considering a 90 degree rotation about the centre of the square. However, to come to this realization and then to formulate the argument would be a significant challenge for most students.

(vi) Here is a standard grade 9 problem: A man is standing in a theatre line. 5/6 of the line is in front of him, and 1/7 of the line is behind. How many people are in the line? This is a standard algebra problem: set up an equation and solve for the variable. But if you think about it for a moment, you realize that if there is an answer, it can be none other that 42.

So I formulated the following problem and gave it to some very good mathematics students: A man is standing in a theatre line. The fraction x of the line is in front of him and the fraction y is behind, where x and y are vulgar fractions written in lowest terms. Show that, if the problem has a solution, then the answer is the least common multiple of the denominators of x and y.

This proved to be a very difficult problem and almost nobody nailed it.

This suggests an area for future analysis in our Study, and we might look at some of the material submitted in this light.

(b) Understanding algorithms: Rote learning has a bad press these days, but handled judiciously, there can be some benefit in students mastering automatic procedures. The students can be challenged in two directions. The first is to carry out some operation is whatever way comes to hand and try to systematize it into a process or procedure that can handle all instances. This for example, occur when the student has solved quadratic equations by means of completing the square and might be asked to devise a formula for solving the general quadratic ax^2 + bx + c = 0. Or the student might have had a lot of experience adding fractions by finding a common denominator and then asked to give a rule for computing a general sum a/b + c/d.

However, we can go in the opposite direction as well, and present the student with a rote procedure and then require him to discover why it works. This is similar to the sort of challenge described in (a), but I present it separately as it deals with algorithms, an important mathematical idea.

(vii) The square root algorithm; Students have become so used to working with calculators that they might not appreciate that there was a paper-and-pencil method of performing this operation that looked somewhat like long division, except that two digits were brought down at a time. I have found that students are interested in this. After working through some examples so that they get the hang of it, they can then be asked to explain why it works.

(viii) The modern student is not generally taught the Euclidean algorithm for find the greatest common divisor of two integers. Thus, one can present the student with a page that describes the algorithm and gives some worked examples. The student is then asked to use it for other pairs of numbers. When it is presented as a rote procedure, students can be encouraged to reflect on whether they find this satisfying. However, there must be some reason that it works, and in trying to find this reason, students are forced to focus on the meaning and properties of greatest common divisor.

(c) Graded challenges: I refer here to problems that can challenge students at different levels, so that there is something for both the weaker and stronger students.

(ix) A mixed grade 3-4 (age 8-9) class was told to make up two whole numbers using each of the digits 1, 2, 3, 4, 5, 6 exactly once and take the difference. Several examples were provided. They were then asked how small they could arrange the difference to be. The class was very adept in the mechanics of subtraction, so it was possible to cover quite a bit of territory in a short time. It became clear that the pupils were at several different levels in engaging the problem, but they all seemed to be involved.

The basic level of challenge is simply to create an example of subtraction, and some students were not able to much beyond this. The second is the realization that, to minimize the difference, one should construct two three-digit numbers (the first example provided has a four-digit minuend and a two-digit subtrahend). The example 642 - 531 came early, which indicates a level of strategy on the part of the proposer. The next level of awareness was that the hundreds digits should differ by 1. Then it was a question of making the subtrahend as large as possible within this constraint and the minuend as small as possible.

There was at least one student who seemed to understand what was going on, because he got the smallest difference (412 - 365 = 47) and stayed there, but he was unable to describe his strategy. However, his friend, sitting next to him, who had not to this point contributed to the discussion, was able to make the explanation.

If challenges are to be a part of regular classroom work, then we need to look at ways in which we can keep all students in the game; it seems to me that there is a shortage of such material.

(x) Another problem that I have used with some success is actually a conjecture due to Paul Erdos: every number of the form 4/n with n >= 3 can be written as the sum of three distinct integer reciprocals. The beauty of this is that all the cases except n one more than a multiple of 24 can be polished off with more or less despatch. If n is even, then students reasonably quickly get town to 4/2m = 2/m = 1/m + 1/m = 1/m + 1/(m+1) + 1/m(m+1). The other cases require a little more ingenuity, but a high school class should get all but the hard case in one or two hours.

(xi) Make a k-digit number whose digits are all distinct, for which the number formed by the leftmost digit is divisible by 1, by the two leftmost digits is divisible by 2, by the three leftmost digits is divisible by 3, and so on. Make k as large as possible. Can you make k equal to 10? (The answer is yes, and the example is unique.)

(c) Investigations: These are challenges that involve students coming up with a plan of action and executing it more or less efficiently.
In these challenges, there may be many solutions. Here are some of this type:

(xii) Let n be a positive integer and let f(n) be the number of times the digit 1 occurs when you write down, in ordinary decimal notation, all the numbers from 1 up to n inclusive. For example, f(12) = 5 since 1 occurs in 1, 10, 11, 12. Is it possible for f(n) to exceed n. When is f(n) = n?

(xiii) In the game frog-jump, you have a row of 5 Xs and 5 Os separate by a blank: X X X X X - O O O O O. The following moves are permitted: (1) any symbol may jump an adjacent symbol into a blank; (2) any symbol may move into an adjacent blank square. Using permitted moves, interchange the positions of the Xs and the Os. The strategy can be built up using few symbols and then the game can be generalized to n symbols of each type.

This might be generalized to the situation that the numbers of the two symbols differ.

(d) Puzzles: There are some nice puzzles that should be given to pupils, if only for cultural reasons (in the same way that one might share a nice piece of music or literature). A lot of these are amenable to group activity.

(xiv) The rotating table: A circular table has four deep wells symmetrically placed, and in each well is a drinking glass. The glasses cannot be seen and one can access them only by sticking one's hands into the wells. Each glass is either upright or inverted, but not all are in the same state. Your task is to get them all into the same state. Here are the rules: The table rotates and stops in a random position. When it does so, you may put your hands into at most two of the wells, feel the tumblers and leave each in whatever state you want. If all tumblers turn out to be in the same state, a buzzer sounds and you know that you have succeeded. Is it possible to *guarantee* success in a finite number of moves? If so, how do you do it?

(xv) The Microsoft problem: Four men are out late at night and have to cross a rickety bridge. The bridge can hold at most two of them and can be crossed only by the light of a single flashlight. So at most two men can cross at one time, and one has to come back with the flashlight, to enable any left behind to cross. The shortest time that each can cross the bridge is 1 minute, 2 minutes, 5 minutes and 10 minutes, and when two cross together they must take the longer time. What is the shortest amount of time required to get all four men across the bridge?

As an extension, suppose the times are a, b, c, d. What is the condition on these numbers that the "obvious" solution is the correct one?

(xvi) A number of coins are strewn on the table, exactly 12 of which are turned up heads and the remainder tails. You are blindfolded, and are required to separate the coins into two piles, each of which has exactly the same number of heads.

These challenges involve students taking note of something that they might have overlooked. It helps for them to realize that there are only a small number of possibilities and so a systematic approach might be suitable.

(xvii) Determine a ten-digit number for which the left digit records the number of 0s, the next digit to the right the number of 1s, the third digit from the left the number of 2s, and so on. This is a nice problem for middle school students; the standard approach is to make a guess (a popular first guess is 900000000); when this does not work, then they students tend to refine this. This strategy will often get to the answer quite quickly. The advantage of this challenge is that the strategy can be systemized to a process of success approximations. Start with any number, and move to a number determined by the number of digits of each kind in the first number; repeat. For example: 3250042578 ---> 2021120110 ---> 3430000000 ---> 7002100000 ---> 7110000100 ---> 6300000100 ---> 7101001000; often this will cycle, but sometimes one gets the answer in this way. This of course, leads to a more extended investigation.

The problem can profitably be generalized; Find all finite sequences (w_0, w_1, ..., w_n) of nonnegative integers for which the number i occurs exactly w_i times, and no integer exceeds n.

(xviii) The problem of the jealous husbands: Three husbands with their wives come to a river that they must cross. All that is available is an old rowboat that can carry at most two people. How can you get everyone across the river in such a way that no woman is in the presence of men other than her husband except when the husband is also there?

This traditional problem offers considerable difficulty to pupils. At each stage, there are not may possibilities - often just one - and the problem seems unsolvable. However, at one point in the solution, one must allow two people to return with the boat rather than just one.

(e) Construction of examples: While this might be subsumed under some of the other categories, I separate it out here because the construction of examples is a more open-ended type of activity, and one that many students are not comfotable with. In constructing a mathematical example, one might try to impose extra conditions that might simplify the situation or try some inspired guesswork; if there are many possibilities, then this activity can be good for a class because one student cannot foreclose the discussion by getting "the answer".

(xix) Use the ten digits, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, each exactly once to construct three whole numbers (involving ten digits in all) for which the largest is the sum of the other two. What are the smallest and largest values of the sum?

(xx) Given four distinct points in the plane, there are six pairs of them, each determining a distance. Generally, one would expect these six distances to be six distinct numbers. However, find examples of configurations for which the six distances are given by only two distinct numbers.

(xxi) Let f(n) be the number of times that the digit 1 occurs when the numbers from 1 up to n inclusive are written down. Thus f(12) = 5 (from the numbers 1, 10, 11, 12). Determine a number n exceeding 1 for which f(n) = n.

(f) Pattern recognition and criterion building: In this, the student investigates a situation and tries to get some insight into the structure of the situation.

(xxii) The reversal of a whole number is obtained by writing the digits backwards. Thus, the reversal of 342 is 243, and the sum of the this number and its reversal is 585. Find a convenient description of those numbers that are the sum of a number and its reversal, when the numbers in the sum have (a) two digits, (b) three digits, (c) four digits.

(xxiii) Here are two numerical equations: $3^2 + 4^2 = 5^2$; $10^2 + 11^2 + 12^2 = 13^2 + 14^2$. Find a conveniently described infinite set of numerical equations that generalize these two.

(g) Analysing fallacies and troubleshooting mistakes: There are many examples of mistakes in mathematics that seem quite plausible at first blush, but which incorporate some error, often quite subtle, but indicating some mathematical detail that might easily be overlooked.
Probably each teacher has faced the situation of marking a solution by a student, realizing that the student is incorrect, but not immediately being able to put a finger on the cause of the mistake.
Giving such mishaps to the students to analyze can be a good challenge that may end up enriching their understanding. Many years ago, I persuaded to editors of the College Mathematics Journal to run a regular feature of these, and over the seventeen or so years that it has been running, we have not been wanting for material, Here is an example:

(xxiv) Since there is a one-one relationship between real numbers and their cubes, the following equations in real variables x and y are equivalent:

(1) $(x+y)^{1/3} + (x-y)^{1/3} = 1$

(2) $2x + 3(x^2 - y^2)^{1/3} = [(x+y)^{1/3} + (x-y)^{1/3}]^3 = 1$

(3) $x^2 - y^2 = ((1-2x)/3)^3$

(4) $y^2 = x^2 - ((1-2x)/3)^3$.

Setting $x = -1$, we obtain from the final equation that $y = 0$. But these values of x and y do not satisfy the first equation.

(h) Technical challenges: The recent trend in avoiding mechanical mathematics is perhaps deserving of reconsideration. In days gone by, it was common to challenge students of algebra and analysis by providing some tricky technical questions (the Cambridge Tripos problems and the contents of the little blue books published a half century ago by the British publisher, Oliver and Boyd, provide good examples of the genre). In high school, it might be a matter of factoring polynomials or solving equations that do not fit into the standard format and require some insight and judicious strategies. In university, such challenges might be found in examples of integration of functions of a single variable, testing series for convergence, evaluating definite integrals in R^2 and

R^3 or using contour integration. The American Mathematical Monthly over the years has supplied a good set of such problems.
Another source of challenges of this type is the old Advanced Level Applied Mathematics course in the United Kingdom.