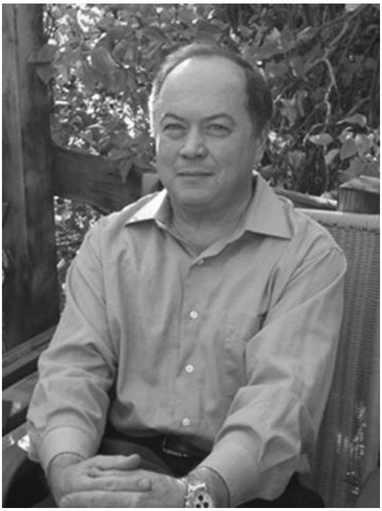
Building a Bridge III: from Problems of Mathematical Olympiads to Open Problems of Mathematics

Alexander Soifer



Born and educated in Moscow, Alexander Soifer has for 29 years been a Professor at the University of Colorado, teaching math, and art and film history. He has written six books and some 200 articles, including The Mathematical Coloring Book, *Springer, September 2008. In fact, Springer has contracted Soifer for seven books, five of which are coming out shortly and two are to be written in 2009–10. He founded and for 25 years ran the Colorado Mathematical Olympiad. Soifer has also served on the Soviet Union Math Olympiad (1970–1973) and USA Math Olympiad (1996–2005). He has been Secretary of WFNMC (1996–present), and a recipient of the Paul Erdös Award (2006). Soifer was the founding editor of the research quarterly* Geombinatorics *(1991–present), whose other editors include Ronald L. Graham, Branko Grünbaum, Heiko Harborth, Peter D. Johnson Jr., and Janos Pach. Paul Erdös was an editor and active author of this journal. Soifer's Erdös number is 1.*

1 Introduction

This is part III of the triptych "Building a Bridge". In part I, I showed an example of how a mathematician can keep an eye open for research ideas that could be used in Mathematical Olympiads. Part II provided an example of a walk in the opposite direction, when an Olympiad problem led to research and problems that are still open today, 20 years later.

The illustration problems in parts I and II had one thing in common: problems were created on one of the shores, and then transplanted across the bridge. I have encountered situations, however, when a problem was conceived on the bridge—and then transplanted to both shores: the shore of olympiads and the shore of mathematical research. I will illustrate such an affair here in the context of a problem I created in early 2004 while at Princeton University.

2 Coffee Hours at Princeton

Coffee pays an important role in mathematics, as Paul Erdős famously observed:

> Mathematician is a machine that converts coffee into theorems.

During the years 2002–2004 I was visiting Princeton University with its fabulous mathematics department, a great fixture of which was a daily 3 to 4 PM coffee hour in the commons room, attended by everyone, from students to the *Beautiful Mind* (John F. Nash Jr.). For one such coffee hour, in February 2004, I came thinking—for the hundredth time in my life—about the network of evenly spaced parallel lines cutting a triangle into small congruent triangles. This time I dealt with equilateral triangles, and the crux of the matter was a demonstration that n^2 unit triangles can cover a triangle of side n. I asked myself a question where the continuous clashes with the discrete: what if we were to enlarge the side length of the large triangle from n to $n+\varepsilon$, how many unit triangles will we need to cover it? This comprised a new open problem:

Cover-Up Problem 1. Find the minimum number of unit equilateral triangles required to cover an equilateral triangle of side $n+\varepsilon$.

During the next coffee hour, I posed the problem to a few Princeton colleagues. The problem immediately excited John H. Conway, the John von Neumann Professor of Mathematics. From the commons room he went to the airport, to fly to a conference. On board the airplane, John found a way (Figure 1) to do the job with just n^2+2 unit triangles! (Area considerations alone show the need for at least n^2+1 of them.) Conway shared his cover-up with me upon his return—at a coffee hour, of course. Now it was my turn to travel to a conference, and have quality time on an airplane. What I found (Figure 2) was a totally different cover-up with the same number, $n^2 + 2$ unit triangles!

Upon my return, at a coffee hour, I shared my cover-up with John Conway. We decided to publish our results together. John suggested setting a new world record in the number of words in a paper, and submitting it to the *American Mathematical Monthly*. On April 28, 2004, at 11:50 AM (computers record the exact time!), I submitted our paper that included just two words, "$n^2 + 2$ can" and our two drawings. I am compelled to reproduce our submission here in its entirety.

Can $n^2 + 1$ unit equilateral triangles cover an equilateral triangle of side $> n$, say $n + \varepsilon$?

John H. Conway & Alexander Soifer

Princeton University, Mathematics, Fine Hall, Princeton, NJ 08544, USA

conway@math.princeton.edu   asoifer@princeton.edu

$n^2 + 2$ can:

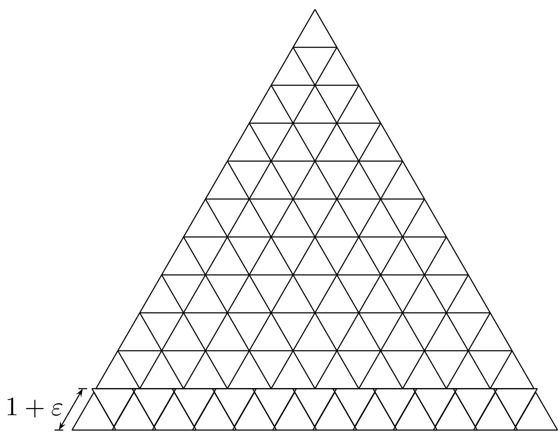


Figure 1

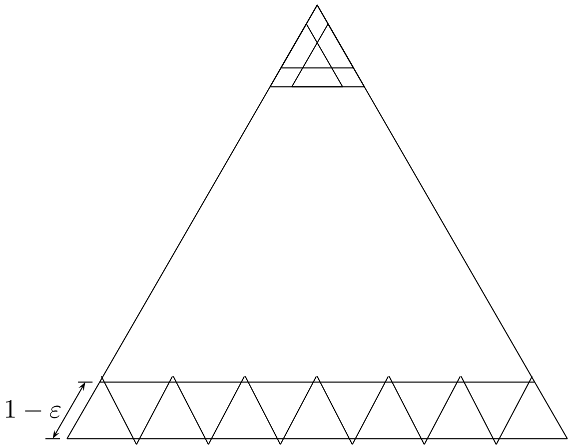


Figure 2

The American Mathematical Monthly was surprised, and did not know what to do about our new world record of a 2-word article. Two days later, on April 30, 2004, the Editorial Assistant Mrs. Margaret Combs acknowledged the receipt of the paper, and continued:

> The Monthly publishes exposition of mathematics at many levels, and it contains articles both long and short. Your article, however, is a bit too short to be a good *Monthly* article... A line or two of explanation would really help.

The same day at the coffee hour I asked John, "What do you think?" His answer was concise, "Do not give up too easily." Accordingly, I replied *The Monthly* the same day:

> I respectfully disagree that a short paper in general—and this paper in particular—merely due to its size must be "a bit too short to be a good *Monthly* article." Is there a connection between quantity and quality?... We have posed a fine (in our opinion) open problem and reported two distinct "behold-style" proofs of our advance on this problem. What else is there to explain?

The Monthly, apparently felt outgunned, for on May 4, 2004, the reply came from *The Monthly*'s top gun, Editor-in-Chief Bruce Palka:

> *The Monthly* publishes two types of papers: "articles," which are substantive expository papers ranging in length from about six to twenty-five pages, and "notes," which are shorter, frequently somewhat more technical pieces (typically in the one-to-five page range). I can send your paper to the notes editor if you wish, but I expect that he'll not be interested in it either because of its length and lack of any substantial accompanying text... The standard way in which we use such short papers these days is as "boxed filler" on pages that would otherwise contain a lot of the blank space that publishers abhor... If you'd allow us to use your paper in that way, I'd be happy to publish it.

John Conway and I accepted the "filler", and in the January 2005 issue our paper [12] was published. *The Monthly*, however, invented the title without any consultation with the authors, and added our title to the body of the article!

We also ran our little article in *Geombinatorics*, where we additionally observed that the "equilaterality" is essential, for otherwise $n^2 + 1$ triangles, similar to the large triangle T and with the ratio of sizes 1: $n + \varepsilon$ can cover T (Figure 3).

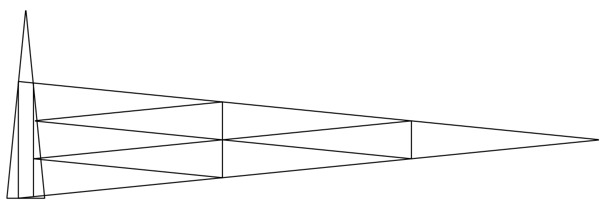


Figure 3

3 Cover-Up at Colorado Mathematical Olympiad

The XXI Colorado Mathematical Olympiad took place shortly after, on April 16, 2004. Needless to say, I wanted to include the Cover-Up problem in some form, preferably as a story.

To Have a Cake (Problem 4, Colorado Mathematical Olympiad)

a) We need to protect from the rain a cake that is in the shape of an equilateral triangle of side 2.1. All we have are identical tiles in the shape of an equilateral triangle of side 1. Find the smallest number of tiles needed.

b) Suppose the cake is in the shape of an equilateral triangle of side 3.1. Will 11 tiles be enough to protect it from the rain?

Solution. 4 a). Mark 6 points in the equilateral triangle of side 2.1: vertices and midpoints (Fig. 4). A tile can cover at most one such point, therefore we need at least 6 tiles. On the other hand, 6 tiles can do

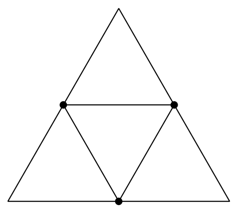


Figure 4

the job. Let me show two different ways, corresponding the general case coverings by Conway and I presented in the previous section. We can first cover the corners (Fig. 5 left), and then use 3 more triangles to cover the remaining hexagon (Fig. 5 right).

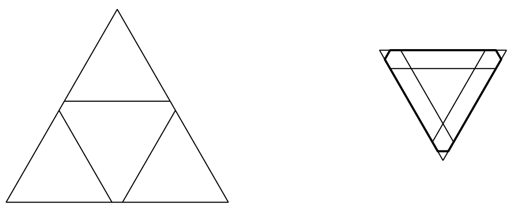


Figure 5

Alternatively, we can cover the top corner (Fig. 6), and then use 5 triangles to cover the remaining trapezoid.

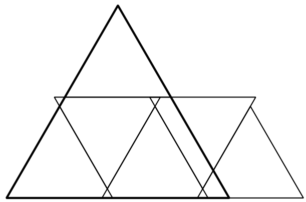


Figure 6

b). We can push the first covering method from a) by covering a cake of side up to 2.25 with 6 tiles (see Fig. 7, where $x = 0.25$).

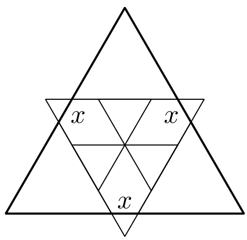


Figure 7

Let us use this to cover a cake of side, say, 2.2 and put this covering in the top corner, and then take care of the remaining trapezoid with 5 tiles (Fig. 8).

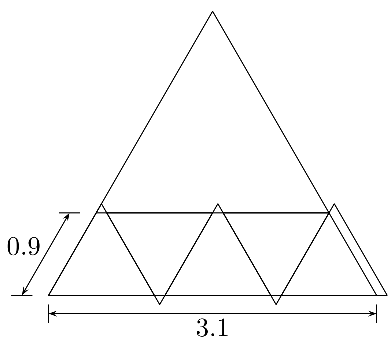


Figure 8

Or we can use 4 tiles in the top corner, and then use 7 tiles for a larger trapezoid (Fig. 9).

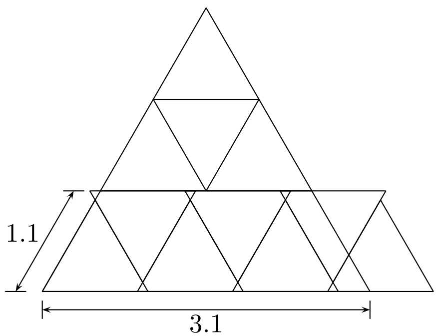


Figure 9

4 Across the Bridge—Back to Research

As I was going across our bridge back to the shore of research, the Columbia University brilliant undergraduate student Mitya Karabash joined me for further explorations of this problem. First of all, we observed the following result, which is better than its simple proof:

Non-Equilateral Cover-Up 2. [14]. For *every* non-equilateral triangle T, n^2+1 triangles similar to T and with the ratio of sizes $1:(n+\varepsilon)$, can cover T.

Proof. An appropriate affine transformation maps the equilateral triangle and its covering depicted in Figure 2 into T. This transformation produces a covering of T with n^2+2 triangles similar to T. We can now cover the top triangle with 2 tiling triangles instead of 3 as shown in Figure 10, thus reducing the total number of covering triangles to n^2+1.

Mitya and I then generalized the problem from covering a triangle to covering much more complex figures we named *trigons.*

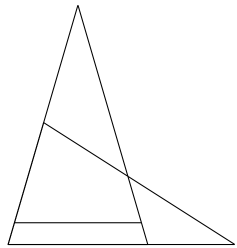


Figure 10

n-Trigon T_n is the union of n triangles from the standard triangulation of the plane such that a triangular rook can find a path between any two triangles of T_n, i.e., the union of n edge-connected triangles. If the triangulation is equilateral, then we say that the n-trigon is *equilateral.* You can see an example of an equilateral 9-trigon in Figure 11.

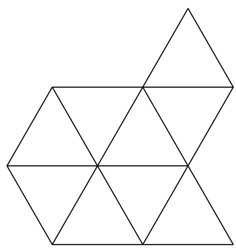


Figure 11

In our cover-up games, we assume that the ratio of the corresponding sides of the triangles forming the trigon and the titling triangles is $(1+\varepsilon)/1$. We proved an important result:

Karabash-Soifer's Trigon Theorem 3. [14]. An n-trigon T_n can be covered with

1. $n+2$ triangles if the trigon is equilateral;
2. $n+1$ triangles if the trigon is non-equilateral.

In spite of all the progress, however, one "little" question remains open. I will formulate it here as a conjecture (because John Conway and I thought we knew the answer—we just had no idea how to prove it):

Cover-Up Conjecture 4 (Conway-Soifer, 2004). Equilateral triangle of side $n+\varepsilon$ cannot be covered by n^2+1 unit equilateral triangles.

Right after the Cover-Up Problem 1, I created the *Cover-Up Squared Problem*. Naturally, a square of side n can be covered by n^2 unit squares. When, however, I let the side length increase merely to $n+\varepsilon$, I found a new open problem:

Cover-Up Squared Problem 5. [13]. Find the smallest number $P(n)$ of unit squares that can cover a square of side length $n+\varepsilon$.

I devised a covering approach illustrated in Figure 12.

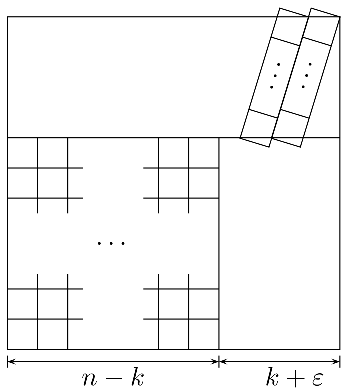


Figure 12

My results were followed by the joint ones by Mitya Karabash and me. The best Mitya and I were able to do in the cover-up squared, was to match Paul Erdős and Ronald L. Graham's dual result [11] on packing squares in a square:

Karabash-Soifer's Theorem 14. [15]. $P(n) = n^2 + O(n^{7/11})$.

Let me explain the "big O" notation. We write $f(n) = O(g(n))$ if asymptotically the function $f(n)$ does not grow faster than a constant multiple of $g(n)$.

The Cover-Up Squared Problem remains open, both in search for the asymptotically lowest possible solution and for exact values for small n. Mitya and I have conjectured:

Cover-Up Square Conjecture 14. [15]. $P(n) = n^2 + \Theta(n^{1/2})$.

We write $f(n) = \Theta(g(n))$ if asymptotically $f(n)$ and $g(n)$ are of the same order, i.e., $f(n) = O(g(n))$ and $g(n) = O(f(n))$.

I hope to have achieved my goal of establishing the bridge between the problems of Mathematical Olympiads and research problems of mathematicians. Paul Erdős told me in March 1989 that "Olympiads by themselves are not very important, but they bring a new enthusiasm for mathematics, and in this regard they are important". I agree. Moreover, building this bridge eliminates any separation between Olympiads and "real" mathematics.

I hope that *Springer*'s latest decision to publish my 7 books [2]–[8] would allow for a further contribution to the Bridge between Olympiads and Mathematics.

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Alexander Soifer
University of Colorado at Colorado Springs
P. O. Box 7150, Colorado Springs, CO 80933
USA
E-mail: asoifer@uccs.edu
http://www.uccs.edu/~asoifer/