Some Problems from Training for a Junior Olympiad

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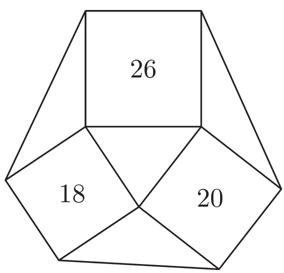
Francisco Bellot Rosado, born in Madrid in 1941, has been chairman of Mathematics at the I.E.S. Emilio Ferrari in Valladolid since 1970. Involved in the preparation of Olympiad students since 1988, he received the Paul Erdös Award from the WFNMC in 2000. He is Western Europe's representative of the WFNMC, and editor of the digital journal Revista Escolar de la O.I.M.

We will present some problems useful in the training of the students who wish take part in a Junior Olympiad, from different sources which are quoted after each statement. We will start with an old but interesting problem from a book of 1917.

Problem 1
The picture shows three squares, of respective areas 26, 18 and 20 square units. Find the area of the hexagon constructed how the picture shows.

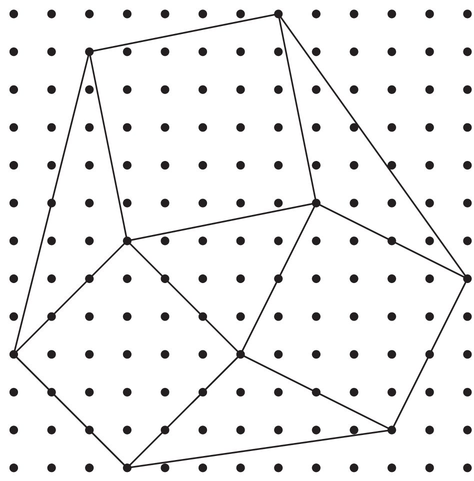


Source: "Amusements in Mathematics", by H. E. Dudeney, First edition 1917.

- The problem has an easy trigonometrical solution but there is a more interesting procedure to find a solution which be suitable for students without trigonometry knowledge.
- This is a versatile problem, in the sense that it is possible to present it to students from 12 years old up to 17 years old.

The key idea is the following:

- It is possible to express 18, 20 and 26 as a sum of two squares?
- $18 = 3^2 + 3^2$; $20 = 4^2 + 2^2$; $26 = 5^2 + 1^2$
- From this, the best is to see a picture below and to count small squares.

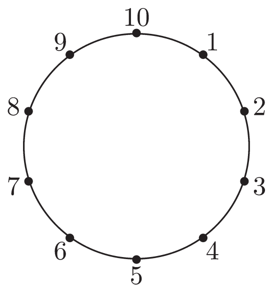


- The area of the hexagon is 100; each of the four triangles are equal to 9.
- The situation can be used as a way to discover Pick's theorem.

A variant of this problem was presented by the late Jim Totten in the Cariboo College of Kamloops (B.C., Canada): if the area of two of the squares equals the area of the rest of the figure, find the measure of one of the angles of the central triangle (the answer is 45°).

Problem 2
Ten people are seated around a round table. Each one thinks of one number and whispers it into the ear of the people seated on his right and left side. Then, every person says out loud the half of the sum of the numbers each one has heard.

The numbers each person says are shown in the picture.



Find the number thought of by the person who says "6".

Source: Hungarian School journal "KöMal" (Közepiskolai Matematikai Lapók)

- Let $x_1, x_2, \ldots, x_{10}$ be the numbers which are thought by the people who said $1, 2, \ldots, 10$ respectively. Immediately we write

$$\begin{aligned} x_1 + x_3 &= 4, & x_2 + x_4 &= 6, \\ x_3 + x_5 &= 8, & x_4 + x_6 &= 10, \\ x_5 + x_7 &= 12, & x_6 + x_8 &= 14, \\ x_7 + x_9 &= 16, & x_8 + x_{10} &= 18, \\ x_9 + x_1 &= 20, & x_{10} + x_2 &= 2. \end{aligned}$$

- If we sum all the equations on the right, we get

$$2(x_2 + x_4 + x_6 + x_8 + x_{10}) = 50$$

  And so $2(6+x_6+18) = 50$, that is, $x_6+24 = 25$ and finally $x_6 = 1$.

- Although the statement does not ask for it, it is possible to find all the numbers.

Problem 3

In a local contest of soccer, only four teams take part: A, B, C, D. Each team played exactly one match against each other. Team A beat team B by 4 goals to 1. Team D beat team C. All the other matches were drawn. The final classification, *by the total of goals scored*, was: 1st A; 2nd B; 3rd C and 4th D. The totals are all different. What was the total of goals in the match C *versus* D?

Source of the problem: Problem set of problems for the Kangaroo Math Contest 2010.

We can start the analysis by means of a table of the goals of each team:

	A	B	C	D
A		4	m	n
B	1		s	t
C	m	s		x
D	n	t	y	

The goals scored by the teams are: A: $4+m+n$; B: $1+s+t$; C: $m+s+x$; D: $n+t+y$

As these goal totals are in decreasing order, the sum of the goals of A and B exceed by at least 4 thee sum of goals of C and D. So, $5-(x+y)$ is at least 4, and this means $x+y=0$ or $x+y=1$. But $y>x$ and then $x=0$, $y=1$.

The goals of A exceed by at least 2 the goals of C, so $4-(s-n)$ is at least 2, or equivalently, $s-n$ is not bigger than 2. Analogously, from the goals of B and D we conclude that $s-n$ is at least 2. Therefore, $s-n=2$.

The number of goals in decreasing order are: $4+m+n$, $3+n+t$, $2+m+n$ and $1+n+t$; from this we conclude $-1<t-m<1$ and so $t=m$.

The table can be rewritten as follows:

	A	B	C	D	Goals in favor	Goals received
A		4	m	n	$4+m+n$	$1+m+n$
B	1		$n+2$	m	$3+m+n$	$6+m+n$
C	m	$n+2$		0	$2+m+n$	$3+m+n$
D	n	m	1		$1+m+n$	$m+n$

So, team D beat team C by $1:0$.

Problem 4
Alex and Petar are at the Riga Station waiting for a train. To make the wait less boring, they decide to play in the following way when a freight train passes on the station, without stopping or reducing the speed. They are together on the platform; when the engine of the freight train reaches the point where they are standing, Alex starts to walk in the same direction of the train, and Petar in the opposite direction, both at the same speed. They stop the moment when the last wagon passes by the point on which they are at that moment. Alex walked 45 meters and Petar 30.

How long is the train?

Source of the problem: Junior Olympiad of Portugal, 2006.

Alex walked $15 = (45-30)$ meters more than Petar, and in the period of time in which Alex walked these 15 meters, the train advanced $45+30 = 75$ meters. Therefore, in the same period of time, the train advanced $75/15 = 5$ times further than each the protagonists. Then, while Petar walked 30 meters, the train advanced $30 \times 5 = 150\,\text{m}$. As Petar starts to walk when the train passes where he was, and stops when the last wagon passes, and walked $30\,\text{m}$ in the opposite way to the train, the train is $150 + 30 = 180\,\text{m}$ long.

Problem 5
Let a, b, c, d be integer numbers and n a positive integer. Suppose that:

1. n divides the sum $a+b+c+d$ and

2. n also divides the sum $a^2+b^2+c^2+d^2$.

Prove that n divides the sum $a^4 + b^4 + c^4 + d^4 + 4abcd$.

Source of the problem: Math. Summer Camp, Zakopane, Poland, presented by Prof. Lev Kurliandchuk, St. Petersburg University

Outline of the solution. The fact that in the statement appears some symmetrical functions of the numbers a, b, c, d suggests to consider the polynomial whose roots are just these numbers:

$$P(x) = (x-a)(x-b)(x-c)(x-d),$$

and using the development of the polynomial and the well-know fact that $P(a) = P(b) = P(c) = P(d) = 0$ we are directed to the solution.

References

[1] *Amusements in Mathematics*, by H. E. Dudeney, First edition 1917; Dover edition 1958.

[2] Hungarian School journal *KöMal* (Közepiskolai Matematikai Lapók), 1995

[3] Problem set of problems for the Kangaroo Math Contest 2010. Minsk (Belarus)

[4] Junior Olympiad of Portugal, 2006.

[5] Math. Summer Camp, Zakopane, Poland 1999, presented by Prof. Lev Kurliandchuk, St. Petersburg University

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